Lecture 17

In which we analyze the zig-zag graph product.

In the previous lecture, we claimed it is possible to "combine" a d-regular graph on D vertices and a D-regular graph on N vertices to obtain a d^2-regular graph on ND vertices which is a good expander if the two starting graphs are. Let the two starting graphs be denoted by H and G respectively. Then, the resulting graph, called the *zig-zag product* of the two graphs is denoted by $G \textcircled{z} H$.

We will use $\lambda(G)$ to denote the eigenvalue with the second-largest absolute value of the normalized adjacency matrix $\frac{1}{d}A_G$ of a d-regular graph G. If $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ are the eigenvalues of the normalized Laplacian of G, then $\lambda(G) = \max\{1 - \lambda_2, \lambda_n - 1\}$.

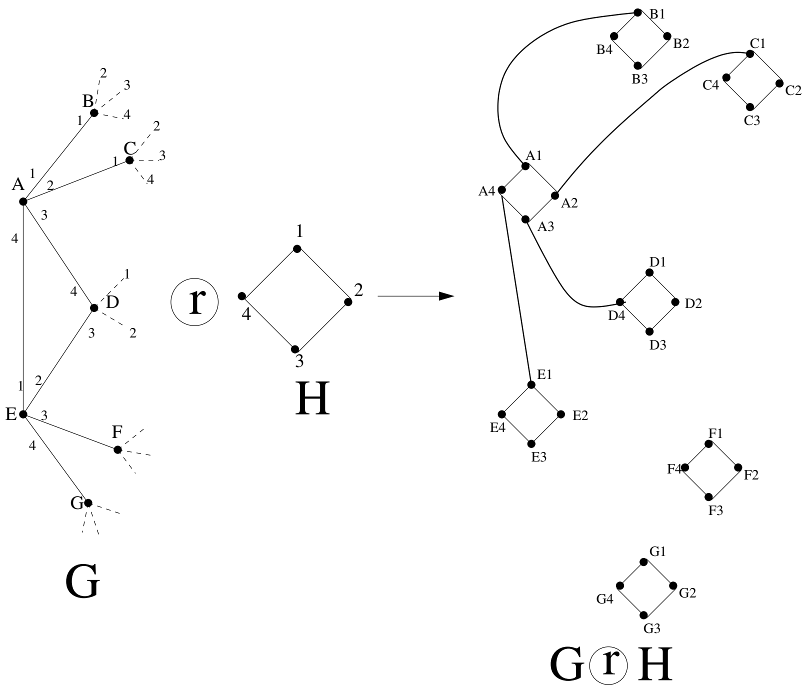
We claimed that if $\lambda(H) \leq b$ and $\lambda(G) \leq a$, then $\lambda(G \textcircled{z} H) \leq a + 2b + b^2$. In this lecture we shall recall the construction for the zig-zag product and prove this claim.

1 Replacement Product and Zig-Zag Product

We first describe a simpler product for a "small" d-regular graph on D vertices (denoted by H) and a "large" D-regular graph on N vertices (denoted by G). Assume that for each vertex of G, there is some ordering on its D neighbors. Then we construct the replacement product (see figure) $G \textcircled{r} H$ as follows:

- Replace each vertex of G with a copy of H (henceforth called a *cloud*). For $v \in V(G), i \in V(H)$, let (v, i) denote the i^{th} vertex in the v^{th} cloud.
- Let $(u, v) \in E(G)$ be such that v is the i-th neighbor of u and u is the j-th neighbor of v. Then $((u, i), (v, j)) \in E(G \textcircled{r} H)$. Also, if $(i, j) \in E(H)$, then $\forall u \in V(G)$ $((u, i), (u, j)) \in E(G \textcircled{r} H)$.

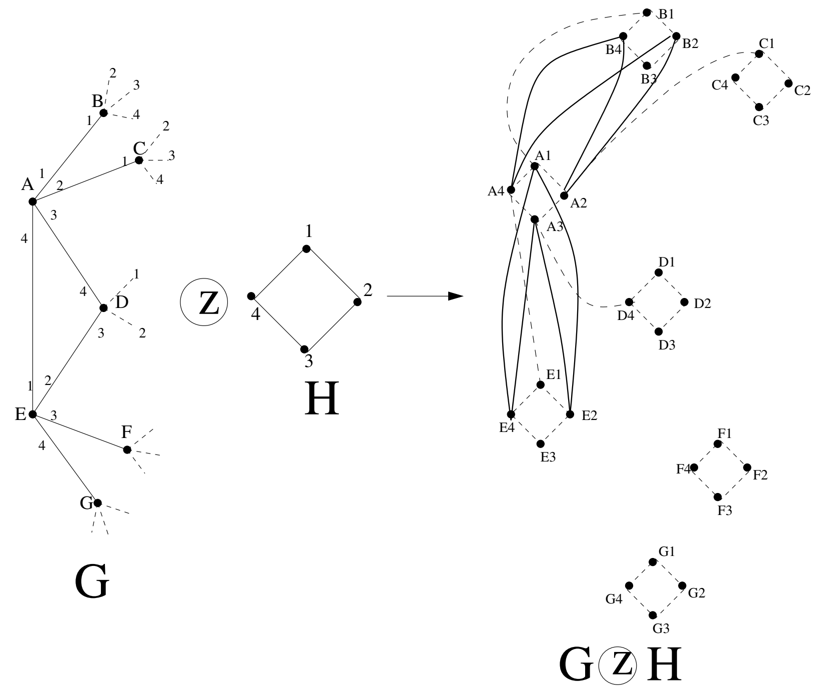
Note that the replacement product constructed as above has ND vertices and is $(d + 1)$-regular.



2 Zig-zag product of two graphs

Given two graphs G and H as above, the zig-zag product $G \textcircled{z} H$ is constructed as follows (see figure):

- The vertex set $V(G \textcircled{z} H)$ is the same as in the case of the replacement product.
- $((u,i),(v,j)) \in E(G \textcircled{z} H)$ if there exist ℓ and k such that $((u,i)(u,\ell)$, $((u,\ell),(v,k))$ and $((v,k),(v,j))$ are in $E(G \textcircled{r} H)$ i.e. (v,j) can be reached from (u,i) by taking a step in the first cloud, then a step between the clouds and then a step in the second cloud (hence the name!).



It is easy to see that the zig-zag product is a d^2-regular graph on ND vertices.

Let $M \in \mathbb{R}^{([N]\times[D])\times([N]\times[D])}$ be the normalized adjacency matrix of $G \textcircled{z} H$. Using the fact that each edge in $G \textcircled{r} H$ is made up of three steps in $G \textcircled{r} H$, we can write M as BAB, where

$$B[(u,i),(v,j)] = \begin{cases} 0 & \text{if } u \neq v \\ \frac{1}{d} & \text{if } u = v \text{ and } \{i,j\} \in H \end{cases}$$

And $A[(u,i),(v,j)] = 1$ if u is the j-th neighbor of v and v is the i-th neighbor of u, and $A[(u,i),(v,j)] = 0$ otherwise.

Note that A is the adjacency matrix for a matching and is hence a permutation matrix.

3 A Technical Preliminary

We will use the following fact. Suppose that $M = \frac{1}{d}A_G$ is the normalized adjacency matrix of a graph G. Thus the largest eigenvalue of M is 1, with eigenvector $\mathbf{1}$; we have

$$\lambda(G) = \max_{\mathbf{x}\perp\mathbf{1}} \frac{|\mathbf{x}^T M \mathbf{x}|}{||\mathbf{x}||^2} = \max_{\mathbf{x}\perp\mathbf{1}} \frac{||M\mathbf{x}||}{||\mathbf{x}||} \tag{1}$$

which is a corollary of the following more general result. Recall that a vector space $S \subseteq \mathbb{R}^n$ is an invariant subspace for a matrix $M \in \mathbb{R}^{n\times n}$ if $M\mathbf{x} \in S$ for every $\mathbf{x} \in S$.

Lemma 1 *Let M be a symmetric matrix, and S be a k-dimensional invariant subspace for M. Thus, (from the proof of the spectral theorem) we have that S has an orthonormal basis of eigenvectors; let $\lambda_1 \leq \cdots \leq \lambda_k$ be the corresponding eigenvalues with multiplicities; we have*

$$\max_{i=1,\dots k} |\lambda_i| = \max_{\mathbf{x}\in S} \frac{|\mathbf{x}^T M\mathbf{x}|}{||\mathbf{x}||^2} = \max_{\mathbf{x}\in S} \frac{||M\mathbf{x}||}{||\mathbf{x}||}$$

Proof: If the largest eigenvalue in absolute value is λ_k, then

$$\max_{i=1,\dots k} |\lambda_i| = \lambda_k = \max_{\mathbf{x}\in S} \frac{\mathbf{x}^T M\mathbf{x}}{||\mathbf{x}||^2}$$

and if it is $-\lambda_1$ (because λ_1 is negative, and $-\lambda_1 > \lambda_n$)

$$\max_{i=1,\dots k} |\lambda_i| = -\lambda_1 = -\min_{\mathbf{x}\in S} \frac{\mathbf{x}^T M\mathbf{x}}{||\mathbf{x}||^2} = \max_{\mathbf{x}\in S} -\frac{\mathbf{x}^T M\mathbf{x}}{||\mathbf{x}||^2}$$

so we have

$$\max_{i=1,\dots k} |\lambda_i| \leq \max_{\mathbf{x}\in S} \frac{|\mathbf{x}^T M\mathbf{x}|}{||\mathbf{x}||^2} \tag{2}$$

From Cauchy-Schwarz, we have

$$|\mathbf{x}^T M\mathbf{x}| \leq ||\mathbf{x}|| \cdot ||M\mathbf{x}||$$

and so

$$\max_{\mathbf{x}\in S} \frac{|\mathbf{x}^T M\mathbf{x}|}{||\mathbf{x}||^2} \leq \max_{\mathbf{x}\in S} \frac{||M\mathbf{x}||}{||\mathbf{x}||} \tag{3}$$

Finally, if $\mathbf{x}_1, \dots, \mathbf{x}_k$ is the basis of orthonormal eigenvectors in S such that $M\mathbf{x}_i = \lambda_i$, then, for every $\mathbf{x} \in S$, we can write $\mathbf{x} = \sum_i a_i \mathbf{x}_i$ and

$$||M\mathbf{x}|| = ||\sum_i \lambda_i a_i \mathbf{x}_i|| = \sqrt{\sum_i \lambda_i^2 a_i^2} \leq \max_{i=1,\dots,k} |\lambda_i| \cdot \sqrt{\sum_i a_i^2} = \max_{i=1,\dots,k} |\lambda_i| \cdot ||\mathbf{x}||$$

so

$$\max_{\mathbf{x}\in S} \frac{||M\mathbf{x}|}{||\mathbf{x}||} \leq \max_{i=1,\dots,k} |\lambda_i| \tag{4}$$

and the Lemma follows by combining (2), (3) and (4). □

4 Analysis of the zig-zag Product

Theorem 2 *Let G be a D-regular graph with n nodes, H be a d-regular graph with D nodes, and let $a := \lambda(G)$, $b := \lambda(H)$, and let the normalized adjacency matrix of $G \textcircled{z} H$ be $M = BAB$ where A and B are as defined in Section 1.*

Then $\lambda(G \textcircled{z} H) \leq a + b + b^2$

Proof: Let $\mathbf{x}\mathbb{R}^{n\times D}$ be such that $\mathbf{x} \perp \mathbf{1}$. We refer to a set of coordinates of $\mathbf{x}$ corresponding to a copy of H as a "block" of coordinate.

We write $\mathbf{x} = \mathbf{x}_{||} + \mathbf{x}_{\perp}$, where $\mathbf{x}_{||}$ is constant within each block, and $\mathbf{x}_{\perp}$ sums to zero within each block. Note both $\mathbf{x}_{||}$ and $\mathbf{x}_{\perp}$ are orthogonal to $\mathbf{1}$, and that they are orthogonal to each other.

We want to prove

$$\frac{|\mathbf{x}^T M \mathbf{x}|}{||\mathbf{x}||^2} \leq a + b + b^2 \tag{5}$$

We have (using the fact that M is symmetric)

$$|\mathbf{x}^T M \mathbf{x}| =\leq |\mathbf{x}_{||}^T M \mathbf{x}_{||}| + 2|\mathbf{x}_{||}^T M \mathbf{x}_{\perp}| + |\mathbf{x}_{\perp}^T M \mathbf{x}_{\perp}|$$

And it remains to bound the three terms.

1. $|\mathbf{x}_{||}^T M \mathbf{x}_{||}| \leq a||\mathbf{x}_{||}||^2$

   Because, after writing $M = BAB$, we see that $B\mathbf{x}_{||} = \mathbf{x}_{||}$, because B is the same as $I_n \otimes \left(\frac{1}{d} A_H\right)$, the tensor product of the identity and of the normalized adjacency matrix of H. The normalized adjacency matrix of H leaves a vector parallel to all-ones unchanged, and so B leaves every vector that is constant in each block unchanged.

   Thus

   $$|\mathbf{x}_{||}^T M \mathbf{x}_{||}| = |\mathbf{x}_{||}^T A \mathbf{x}_{||}$$

   Let $\mathbf{y}$ be the vector such that y_v is equal to the value that $\mathbf{x}_{||}$ has in the block of v. Then

   $$|\mathbf{x}_{||}^T A \mathbf{x}_{||}| = 2 \sum_{\{(v,i),(w,j)\} \in E_{G \textcircled{z} H}} y_v y_w = \mathbf{y}^T A_G \mathbf{y} = aD||\mathbf{y}||^2 \leq a||\mathbf{x}_{||}||^2$$

   because $\mathbf{y} \perp \mathbf{1}$ and $||\mathbf{y}||^2 = \frac{1}{D}||\mathbf{x}_{||}||^2$

2. $|\mathbf{x}_\perp^T M \mathbf{x}_\perp| \leq b^2 ||\mathbf{x}_\perp||^2$ Because, from Cauchy-Schwarz and the fact that permutation matrices preserve length, we have

$$|\mathbf{x}_\perp^T BAB\mathbf{x}_\perp| \leq ||B\mathbf{x}_\perp|| \cdot ||AB\mathbf{x}_\perp|| = ||B\mathbf{x}_\perp||^2$$

Now let us call $\mathbf{x}_\perp^v$ the restriction of $\mathbf{x}_\perp$ to coordinates of the form (v,i) for $i = 1,\ldots,D$. Then each $\mathbf{x}_\perp^v$ is orthogonal to the all-one vector and $A_H \mathbf{x}_\perp^v \leq db||\mathbf{x}_\perp^v||$, so

$$||B\mathbf{x}_\perp||^2 = \sum_v ||d^{-1} A_H \mathbf{x}_\perp^v||^2 \leq \sum_v b^2 ||\mathbf{x}_\perp^v||^2 = b^2 ||\mathbf{x}_\perp||^2$$

3. $2|\mathbf{x}_{||}^T M \mathbf{x}_\perp| \leq b||\mathbf{x}||^2$

Because, from Cauchy-Schwarz, the fact that $B\mathbf{x}_{||} = \mathbf{x}_{||}$ and the fact that permutation matrices preserve length, we have

$$|\mathbf{x}_{||}^T BAB\mathbf{x}_\perp| \leq ||B\mathbf{x}_{||}|| \cdot ||AB\mathbf{x}_\perp|| = ||\mathbf{x}_{||}|| \cdot ||B\mathbf{x}_\perp||$$

and we proved above that

$$||B\mathbf{x}_\perp|| \leq b||\mathbf{x}_\perp||$$

so

$$|\mathbf{x}_{||}^T BAB\mathbf{x}_\perp| \leq b \cdot ||\mathbf{x}_{||}|| \cdot ||\mathbf{x}_\perp|| \leq \frac{b}{2}(||\mathbf{x}_{||}||^2 + ||\mathbf{x}_\perp||^2) = \frac{b}{2}||\mathbf{x}||^2$$

□