Problem Set 1

Solution

As in the textbook, H is the 1-qubit unitary transformation

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

1. [30/100] Let $f : \{0,1\}^2 \to \{0,1\}^2$ be the function that adds 1 modulo 4 to the input (both the input and the output are binary representations of integers in the set $\{0,1,2,3\}$), thus $f(00) = 01$, $f(01) = 10$, $f(10) = 11$, $f(11) = 00$, and let U_f be the 2-qubit unitary transformation corresponding to f.

   Consider the following quantum process: starting from the state $|00\rangle$, first apply H to the second qubit, and then apply U_f to the resulting quantum state. Finally, measure the first bit.

   What is the quantum state before the measurement? Are the two qubits entangled? What are the probabilities that the outcome of the measurement is zero and that the outcome of the measurement is one? What is the residual quantum state in each case?

   ## Solution:

   The quantum state before measurement is

   $$U_f[I \otimes H]|00\rangle = U_f\left[\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle\right] = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle.$$

   The two qubits are entangled, since the state of the system cannot be decomposed as a product of two component states. The probability that the outcome of the measurement is 0 is

   $$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2},$$

and the residual state in this case is

$$\frac{\frac{1}{\sqrt{2}}|01\rangle}{\frac{1}{\sqrt{2}}} = |01\rangle$$

Similarly, probability that the outcome of the measurement is 1 is 1/2, and the residual state in this case is $|10\rangle$.

2. [35/100] Consider the following quantum process involving n-qubit strings. Start from the all-ones classical state $|11\cdots 1\rangle \in \mathbb{C}^{\{0,1\}^n}$, and then, for $i = 1, \ldots, n$, apply H to the i-th qubit. Finally, measure the first $n-1$ bits. What are the probabilities of the various possible outcomes? What is the residual quantum state in each case?

Solution:

Observe that for $x \in \{0,1\}$, we have

$$H|x\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}(-1)^x|1\rangle,$$

which can be also written as

$$H|x\rangle = \frac{1}{\sqrt{2}} \sum_{y\in\{0,1\}} (-1)^{xy}|y\rangle.$$

In general, for every $x \in \{0,1\}^n$,

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x_1y_1+\cdots+x_ny_n}|y\rangle.$$

Starting from the state $|11\cdots 1\rangle$, after applying H to each qubit, we get

$$H^{\otimes n}|11\cdots 1\rangle = \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{y_1+\cdots+y_n}|y\rangle$$

If we measure the first $n-1$ qubits, the outcomes are all possible $n-1$ bit strings, all equally likely, each with probability $\frac{1}{2^{n-1}}$. If the outcome of the measurement is the bit string $x_1, ..., x_{n-1}$, then the residual state is

$$\begin{aligned} & |x_1, ..., x_{n-1}\rangle \otimes \frac{\frac{1}{\sqrt{2^n}} \sum_{w\in\{0,1\}} (-1)^{x_1+...+x_{n-1}+w}|w\rangle}{\frac{1}{\sqrt{2^{n-1}}}} \\ = \; & |x_1, ..., x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}(-1)^{x_1+...+x_{n-1}}\big(|0\rangle - |1\rangle\big). \end{aligned}$$

3. [35/100] (Based on Exercise 4.35 from the textbook.) If $U \in \mathbb{C}^{\{0,1\}\times\{0,1\}}$ is a 1-qubit unitary transformation, then controlled-U is the 2-qubit unitary transformation $C_U \in \mathbb{C}^{\{0,1\}^2\times\{0,1\}^2}$ such that for every bit $b \in \{0,1\}$ we have $C_U|0b\rangle = |0b\rangle$ and $C_U|1b\rangle = |1\rangle \otimes U|b\rangle$.

   (a) Write out C_H explicitly as a 4×4 matrix

   (b) Let $q \in \mathbb{C}^{\{0,1\}^2}$ be a 2-qubit pure quantum state. Consider the following two processes: (A) we measure the first bit of q and then, if the outcome of the measurement is 1 we apply H to the second bit, otherwise we do nothing; (B) we apply C_H to q and then we measure the first bit. Prove that process A and process B are the same, that is, that for every initial pure quantum state $q = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ the outcome of the measurement has the same distribution and, for each outcome, the residual final quantum state is also the same.

Solution:

(a)

$$C_H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

(b)
Initial quantum state:

$$q = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

In process (A), if we measure the first bit of q, it has the following possible outcomes:

• 0 with probability $p_0 := |a_{00}|^2 + |a_{01}|^2$, in which case the residual state is

$$\frac{(a_{00}|00\rangle + a_{01}|01\rangle)}{\sqrt{p_0}}$$

which is also the final state, since we do nothing.
• 1 with probability $p_1 := |a_{10}|^2 + |a_{11}|^2$, in which case the residual state is

$$\frac{(a_{10}|10\rangle + a_{11}|11\rangle)}{\sqrt{p_1}},$$

and after the application of H to the second bit, the final state is

$$\frac{(a_{10}|10\rangle + a_{10}|11\rangle + a_{11}|10\rangle - a_{11}|11\rangle)}{\sqrt{2}\sqrt{p_1}}.$$

In process (B), the application of C_H produces the quantum state

$$a_{00}|00\rangle + a_{01}|01\rangle + \frac{1}{\sqrt{2}}\Big(a_{10}|10\rangle + a_{10}|11\rangle + a_{11}|10\rangle - a_{11}|11\rangle\Big)$$

and the measurement of the first bit has the following outcomes:

• 0 with probability $|a_{00}|^2 + |a_{01}|^2 = p_0$, in which case the residual state is

$$\frac{a_{00}|00\rangle + a_{01}|01\rangle}{\sqrt{p_0}}$$

• 1 with probability $\frac{|a_{10}|^2+|a_{10}|^2+|a_{11}|^2+|-a_{11}|^2}{2} = |a_{10}|^2 + |a_{11}|^2 = p_1$, in which case the residual state is

$$\frac{a_{10}|10\rangle + a_{10}|11\rangle + a_{11}|10\rangle - a_{11}|11\rangle}{\sqrt{2}\sqrt{p_1}}$$

Thus, Process A and Process B are the same.