Problem Set 5

Updated Dec 2, 2012: in Problem 3 the density matrix is assumed to have real entries.

This problem set is due on Thursday, December 6.

1. 10/100 Compute the density matrix of the following mixed one-qubit quantum state: with probability 1/2 we have $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and with probability 1/2 we have $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

2. 20/100 Find an operator sum representation of the following process: given a two-qubit state, apply the Hadamard transform to the first qubit, then measure the second qubit.

3. 20/100 Let us denote by $\{(p_1, |q_1\rangle), \ldots, (p_k, |q_k\rangle)\}$ a mixed quantum state in which the pure quantum state $|q_i\rangle$ occurs with probability p_i.

   If $\{(p_1, |q_1\rangle), \ldots, (p_k, |q_k\rangle)\}$ and $\{(p'_1, |q'_1\rangle), \ldots, (p'_h, |q'_h\rangle)\}$ are two mixed quantum states with the same density matrix

   $$M = \sum_{i=1}^{k} p_i |q_i\rangle\langle q_i| = \sum_{i=1}^{h} p'_i |q'_i\rangle\langle q'_i|$$

   then the outcome of any experiment (where by experiment we mean a unitary transformation followed by a, possibly partial, measurement) has the same distribution in both states.

   Prove that the converse is also true for one-qubit states whose density matrices has all real entries (it is true for any number of qubits and complex entries – the restrictions are to make the problem easier), that is, prove that if $\{(p_1, |q_1\rangle), \ldots, (p_k, |q_k\rangle)\}$ and $\{(p'_1, |q'_1\rangle), \ldots, (p'_h, |q'_h\rangle)\}$ are two one-qubit mixed quantum states with different real density matrices

   $$\sum_{i=1}^{k} p_i |q_i\rangle\langle q_i| \neq \sum_{i=1}^{h} p'_i |q'_i\rangle\langle q'_i|$$

Then there is an experiment whose outcome has a different distribution for the first mixed state than for the second mixed state.

[Hint: either just measure the state or apply a Hadamard transform and then measure the state]

4. 50/100 Consider the code

$$\begin{aligned} |0\rangle &\rightarrow \frac{1}{\sqrt{8}}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) \\ |1\rangle &\rightarrow \frac{1}{\sqrt{8}}(|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \otimes (|0\rangle - |1\rangle) \end{aligned}$$

which protects against at most one phase flip error.

Prove that the code does not protect against one bit flip error.

[Hints: call $E(|q\rangle)$ the three-qubit encoding of the 1-qubit quantum state q. The fact that the code protects from phase flip errors implies that for every two different one-qubit quantum states $|q_1\rangle$ and $|q_2\rangle$ and for every two arbitrary three qubits channels C_1 and C_2 (where C_1 and C_2 are both unitary operators which are either the identity or the tensor of two one-qubit identity matrix and a phase flip matrix) we have that $C_1 \cdot E(|q_1\rangle) \neq C_2 \cdot E(|q_2\rangle)$.

To prove that correction of one bit flip error is not always possible, you have to show that there are two different quantum states $|q_1\rangle$ and $|q_2\rangle$, and two channels C_1 and C_2 (each being either the tensor of two one-qubit identity matrices and one bit-flip matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or the 3-qubit identity matrix) such that $C_1 \cdot E(|q_1\rangle) \neq C_2 \cdot E(|q_2\rangle)$.

It might be helpful to first think about the (closely related) problem of showing that the code $|0\rangle \rightarrow |000\rangle$, $|1\rangle \rightarrow |111\rangle$ does not protect against one phase flip error.]