Problem Set 1

This problem set is due on Thursday, October 11, by 2:15pm. You can either hand it in class or email a pdf to Joongyeub.

As in the textbook, H is the 1-qubit unitary transformation

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

1. [30/100] Let $f : \{0,1\}^2 \to \{0,1\}^2$ be the function that adds 1 modulo 4 to the input (both the input and the output are binary representations of integers in the set $\{0,1,2,3\}$), thus $f(00) = 01$, $f(01) = 10$, $f(10) = 11$, $f(11) = 00$, and let U_f be the 2-qubit unitary transformation corresponding to f.

   Consider the following quantum process: starting from the state $|00\rangle$, first apply H to the second qubit, and then apply U_f to the resulting quantum state. Finally, measure the first bit.

   What is the quantum state before the measurement? Are the two qubits entangled? What are the probabilities that the outcome of the measurement is zero and that the outcome of the measurement is one? What is the residual quantum state in each case?

2. [35/100] Consider the following quantum process involving n-qubit strings. Start from the all-ones classical state $|11\cdots 1\rangle \in \mathbb{C}^{\{0,1\}^n}$, and then, for $i = 1, \ldots, n$, apply H to the i-th qubit. Finally, measure the first $n-1$ bits. What are the probabilities of the various possible outcomes? What is the residual quantum state in each case?

3. [35/100] (Based on Exercise 4.35 from the textbook.) If $U \in \mathbb{C}^{\{0,1\}\times\{0,1\}}$ is a 1-qubit unitary transformation, then controlled-U is the 2-qubit unitary transformation $C_U \in \mathbb{C}^{\{0,1\}^2\times\{0,1\}^2}$ such that for every bit $b \in \{0,1\}$ we have $C_U|0b\rangle = |0b\rangle$ and $C_U|1b\rangle = |1\rangle \otimes U|b\rangle$.

   (a) Write out C_H explicitly as a 4×4 matrix

(b) Let $q \in \mathbb{C}^{\{0,1\}^2}$ be a 2-qubit pure quantum state. Consider the following two processes: (A) we measure the first bit of q and then, if the outcome of the measurement is 1 we apply H to the second bit, otherwise we do nothing; (B) we apply C_H to q and then we measure the first bit. Prove that process A and process B are the same, that is, that for every initial pure quantum state $q = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$ the outcome of the measurement has the same distribution and, for each outcome, the residual final quantum state is also the same.