Problem 1

Assume for the sake of contradiction that R recognizes L.
Let X_i be all strings of length i in lexi order. Formally, $X_i = \text{sort}\{s \mid |s| = n\}$.
Let c_1 be some constant such that forall s, $K(s) \leq |s| + c_1$.[1]

```
def more_complex_than (r, s) :: String -> String -> Bool
  if( R(r,s) == ACCEPT) return ACCEPT

def more_complex_than_set (r, S) :: String -> Set of Strings -> Bool
  for all s in S:
    fork more_complex_than (r, s)
  if (all threads ACCEPT) return ACCEPT

def find_incompressible_string (n) :: Int -> String
  for every s in X_(n+c_1+1):
    fork more_complex_than_set (s, X_n)
  ans_n = first s to be accepted
  return ans_n
```

Let c_2 be the size of the program above. We now claim that for all n, we have $2 * c_2 + 3 + \log n \geq K(ans_n) \geq n$, which is impossible for sufficiently large n.

- First, we prove that ans_n must exist:
  - By textbook lemma, we know $\exists s \in X_{n+c+1} : K(s) \geq |s| = n + c + 1$.
  - By textbook lemma, we also know $\forall s \in X_n : K(s) \leq n + c$.
  - Thus, there is some string in X_{n+c+1} which is more complex than all strings in X_n, and thus ans_n must exist (because all of its calls to R must termiante and accept).
- Next, we will prove that $2 * c_2 + 3 \log n \geq K(ans_n)$. For this, consider $\underbrace{bd(Prog)}_{\leq 2c_2}\underbrace{01}_{2}\underbrace{n}_{\leq 1+\log n}$
- Lastly, we prove that $K(ans_n) \geq n$. This follows from: $K(ans_n) \geq \max_{s \in X_n} K(s) \geq n$.

[1]this exists by textbook lemma

Problem 2

Our general idea is:

1. Assume for the sake of contradiction that R recognizes L.
2. Using R, show that $S = \{x \mid K(x) \geq |x|\}$ is recognizable.
3. Show this leads to a contradiction.

Statement 1 $\Rightarrow$ Statement 2

We define R_S, a recognizer for S, as follows:

```
R_S(x):
  run R(x, x, |x|-1)
  accept iff R accepts
```

Statement 2 $\Rightarrow$ Statement 3

```
M(n):
  for s <- sort { all strings of length n }
    run R_S(s)
  ans_n = first s to be accepted
  return ans_n
```

As in problem 1, we now get $2 * c_2 + 3 + \log n \geq K(ans_n) \geq n$, contradiction.

Problem 3

Suppose for the sake of contradiction that f is unbounded.

```
def S(n):
  for x <- Sigma^* in lexi order
    if (f(x) >= n):
      return x
```

We note that by construction, $K(S(n)) \geq f(S(n)) \geq n$. However, it also has a description $\underbrace{bd(S)}_{2c_2}\underbrace{01}_{2}\underbrace{n}_{1+\log n}$ of size $2 * c_2 + 3 + \log n$, contradiction.