Problem 1

- **Solution 1**

  $\{L \mid L \subseteq \Sigma^*\}$ is uncountable. The number of TMs is countable. Therefore, there is an undecidable subset of Σ^*.

- **Solution 2**

  Consider $L = \{1^{E\langle M,x\rangle} \mid M(x) \text{ halts }\}$ for some encoding E (for example, $E(x, y) = bin(1 \circ bd(x) \circ 01 \circ y))$.

Problem 2

- L has recognizer $R \Rightarrow$ exists decider D s.t. $L = \{x \mid \exists y : D(x, y) \text{ accepts }\}$:

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D(x,y) = does R(x) accept within y steps?
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- exists decider D s.t. $L = \{x \mid \exists y : D(x, y) \text{ accepts }\} \Rightarrow L$ has recognizer R:

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R(x) =
  for y in Sigma^* in lexi order:
    if D(x,y) accepts, accept
```

Problem 3

- **L is countable**
  - Let $L_M = \{x \mid M(x) \in A\}$
  - Let $L' = \{L_M \mid M \text{ is a TM }\}$.
  - Note that L' is countable since the number of TMs is countable. We will now show that $L \subseteq L'$.
  - Take any $B \in L$, there must be some M_B such that $x \in B \Leftrightarrow M_B(x) \in A$. Thus, $B = \{x \mid M_B(x) \in A\} = L_{M_B} \in L'$.
- **U is uncountable**
  - Consider $f(x) = 0 \circ x$.
  - For a string x and a set S, define $x \circ S = \{x \circ y \mid y \in S\}$.
  - Consider $S = \{(0 \circ A) \cup (1 \circ X) \mid X \subseteq \Sigma^*\}$.
  - It follows that $\forall B \in S : \forall x : x \in A \Leftrightarrow f(x) \in B$. Thus, $S \subseteq U$.
  - Furthermore, note that since $|S| = |2^{\Sigma^*}|$, S is uncountable.

Grading Rubric

- There's basically 5 proofs: P1 (30), P2 one direction (15), P2 other direction (15), P3 countable (20), P3 uncountable (20).
- For each section:
  - Decide if solution is "basically correct" or "way off" (incorrect reduction; reducing in wrong direction; etc ...)
  - "Way off" solutions = 0 points
  - "Basically correct solutions" = start from full credit, deduct points as necessary for minor technical mistakes
  - When taking off points, provide a short (1-2 sentence) explaination for why points are being deducted.