Problem 1 Solution

To show ShortestPath$\in$NL, we'll first consider nondeterministic logspace machine that given (G, s, t, k) will decide if there is a path from s to t is *less than or equal to* k length thus showing this more general problem is in NL:

Input: $(G = (V, E), s, t, k)$
  Let $p = s$
  for $i = 1, ..., k$
    if $p == t$ accept
    Guess $p =$neighbor of p
  reject

So, deciding if there's a $\leq k$ length path from s to t is in NL. And since NL=coNL, then complement problem of deciding if there does not exist a an $s - t$ path of length $\leq k$ is NL. Moreover, there's a shortest path of length exactly k iff there's a path of length $\leq k$ and there's not a path of length $\leq k - 1$, both of which have a logspace nondeterminstic machine deciding them. And so ShortestPath must also be in NL.

And to show ShortestPath$\in$NL-hard we will show $\overline{\text{ST-REACH}} \leq_m^L$ShortestPath since NL=coNL. That is, we want a log-space transducer that takes input $G = (V, E), s, t$ and outputs $G' = (V', E'), s', t', k'$ such that

$$(G, s, t) \in \overline{\text{ST-REACH}} \Leftrightarrow (G', s', t', k') \in \text{ShortestPath}$$

To do this let's take (G, s, t) and preserve s, and t and change G into G' by letting V' be V with $n = |V|$ extra vertices and by letting E' be E with the extra edges necessary to have a direct path from s to t through only and all of the newly introduced n vertices. So now G' is a copy of G with one extra overarching path from s to t of length $n + 1$. Finally, let k' be $n + 1$. We just need to argue our iff statement holds.

If $(G, s, t) \in \overline{\text{ST-REACH}}$, then there is not any path from s to t and so the only path in G' is the $n + 1$ length path we added and so $(G', s', t', n + 1) \in$ ShortestPath. And if $(G', s', t', n + 1) \in$ ShortestPath, then the shortest path in G' is length $n+1$ and so that must have been the path we added and couldn't have been in G since there were only n vertices in G. Furthermore, there is no other path in G' since $n+1$ is the shortest and so G couldn't have had any $s-t$ path and thus $(G, s, t) \in \overline{\text{ST-REACH}}$. So $\overline{\text{ST-REACH}} \leq_m^L$ShortestPath and so ShortestPath$\in$NL-hard.

Problem 2 Solution

Suppose $A \in$SPACE(n^2). This implies that we have a machine, M_A, that takes input x and decides if $x \in A$ in space $|x|^2$. Now consider the polynomial time reduction that takes x and turns it into the string $x \circ 0^{|x|^2}$ - i.e. x padded with $|x|^2$ zeroes. Since we're just adding a polynomial number of zeroes, this is certainly a polynomial time reduction.

Now consider the machine that takes this new input and just looks at the first $|x|$ bits and then runs M_A on them. M_A will take $|x|^2$ space to decide if $x \in A$ but $|x|^2$ is linear on the size of the padded input! So this machine must also decide A but in space linear to the size of its input. Thus, $A \in$SPACE(n) and, since A was arbitrary, SPACE$(n^2)\subseteq$SPACE(n). But this contradicts the space hierarchy theorem since SPACE$(n) subset$SPACE(n^2). So it can't be the case the SPACE(n) is closer under polynomial time reductions.

Problem 3 Solution

If NP=SPACE(n^2), then SPACE(n^2) would be closed under polynomial time reductions since NP is. However, we can make an almost identical argument to what we did in problem 2 to show that SPACE(n^2) is not closer under polynomial time reductions. So NP can't equal SPACE(n^2).