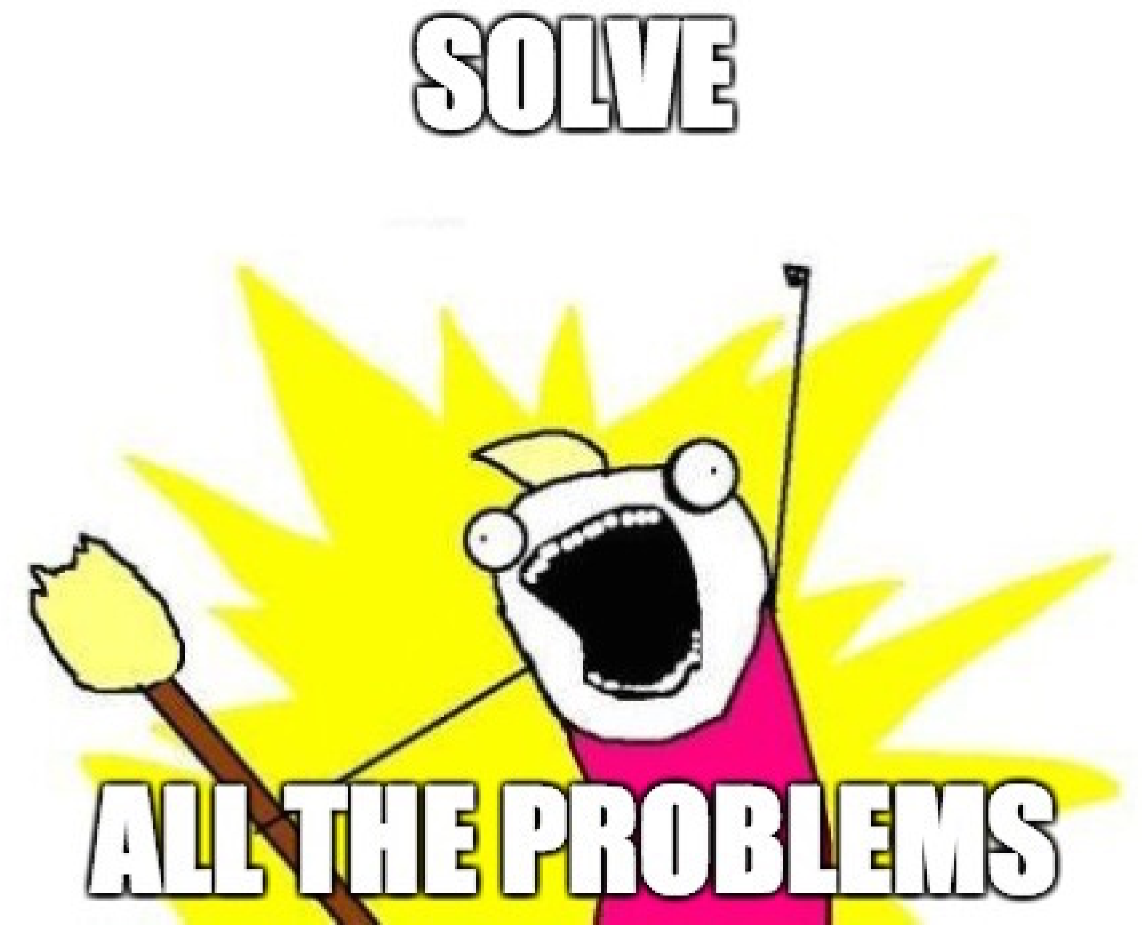
Practice Midterm



- Don't panic.
- The practice midterm has 9 problems.
- The real midterm has 3 problems.

Problem 1

Let $\Sigma = \{0, 1\}$. Let `reverse` $: \Sigma^* \to \Sigma^*$ be the reverse function. Prove that for every k, there exists a language L where the minimal DFA for L has $k + 2$ states, but the minimal DFA for $L^R = \{\texttt{reverse}(x) \mid x \in L\}$ has 2^k states.

Given k, define L_k

Prove L_k has exactly $k + 2$ equivalence classes.

Prove L_k^R has exactly 2^k equivalence classes.

Problem 2

Let $\Sigma = \{0, 1\}$. Let $\circ$ be the concat operator.
Prove there exists some regular language L such that $L' = \{w \circ w | w \in L\}$ is irregular.

Define L

Argue that L is regular

Show that L' has an infinite number of equivalence classes

Problem 3

Let $\Sigma = \{0, 1\}$. Let $\circ$ be the concat operator.
Prove that for every regular L, $L' = \{w | w \circ w \in L\}$ is regular.

Let L be any regular language. Show that L' is also regular.

Hint, take the DFA for L, construct a DFA for L'

Problem 4

Let $\Sigma = \{0, 1\}$. For a NFA, an accepting path is any path from the starting state to any accepting state, consistent with the input. A 2NFA is an ϵ-free NFA that accepts if and only if the total number of accepting paths is exactly 2.
Prove that 2NFAs only recognize regular languages.

Let L be any language recognized by a 2NFA. Prove L is regular.

Hint: construct a DFA

Problem 5

Let $\Sigma = \{0, 1\}$. For a NFA, an accepting path is any path from the starting state to any accepting state, consistent with the input. An odd-NFA is an ϵ-free NFA that accepts if and only if the total number of accepting paths is odd.
Prove that odd-NFAs only recognize regular languages.

Let L be any language recognized by an odd-NFA. Prove L is regular.

Hint: construct a DFA

Problem 6

Let $\Sigma = \{0, 1\}$. The Turing Machines we have seen in class are 1D-TM since they operate on a 1D-tape. A 2D-TM operates on a 2D-grid and has transition function $\delta : Q \times \Gamma \rightarrow Q \times \Gamma \times \{LEFT, RIGHT, UP, DOWN\}$. (The input is written along the x-axis). Prove that a 2DTM is equivalent in power to a 1DTM.

Design an algorithm executable on a 1D-TM for simulating 2D-TMs.

Problem 7

Let $\Sigma = \{0, 1\}$. For $s, r \in \Sigma^*$, we say that $s \prec r$ if (1) $|s| < |r|$ or (2) $|s| = |r|$ and s appears ahead of r in alphabetical order.
A sequence of strings $s_1, s_2, ...$ is in lexicographical order if for all $i < j$, we have $s_i \prec s_j$.
A language is lexicographically enumerable if there is an enumerator that outputs the language in lexicographical order.
Prove that a language is lexicographically enumerable if and only if it is decidable.

Assume L is lexicographically enumerable. Prove L is decidable.

Assume L decidable. Prove L is lexicographically enumerable.

Problem 8

For every integer n, consider the streaming complexity of the problem of deciding whether a graph on n vertices, given by a stream of edges, is bipartite.

That is, for a set of n vertices V, our alphabet $\Sigma = \{\{x, y\} \mid x, y \in V, x \neq y\}$ is all possible (undirected) edges between these vertices and our stream is a sequence of these edges. If we call the set of each edge in this stream E, then $G = (V, E)$ is the undirected graph defined by it. We want a streaming algorithm that takes the stream and computes whether or not G is bipartite. Show that every streaming algorithm for this problem requires $\Omega(n)$ bits of memory.

Define Q, the set of distinguishable strings.

Argue that for any $s, r \in Q$, they are distinguishable.

Argue that $|Q| = 2^{\Omega(n)}$

Problem 9

For every integer n, consider the streaming complexity of the problem of deciding whether a graph on n vertices, given by a stream of edges, is connected.

That is, for a set of n vertices V, our alphabet $\Sigma = \{\{x, y\} \mid x, y \in V, x \neq y\}$ is all possible (undirected) edges between these vertices and our stream is a sequence of these edges. If we call the set of each edge in this stream E, then $G = (V, E)$ is the undirected graph defined by it. We want a streaming algorithm that takes the stream and computes whether or not G is connected. Show that every streaming algorithm for this problem requires $\Omega(n \log n)$ bits of memory.

Define Q, the set of distinguishable strings.

Hint: you are not expected to solve this problem. We have put it here in case the rest of the practice midterm was not entertaining.

Argue that for any $s, r \in Q$, they are distinguishable.

Argue that $|Q| = 2^{\Omega(n \log n)}$