Notes for Lecture 7

This lecture is based on the Goemans-Williamson paper [4], and Vazirani's book [13].

Outline

1. **Max-Cut – problem definiton:**

   Given an undirected graph $G = (V, E)$, find a partition of the vertex set $V = S \cup \bar{S}$ that maximizes the number of cut-edges (edges with an endpoint in S and an endpoint in $\bar{S}$).

   Examples: A clique, a bipartite graph, an odd cycle.

   The problem is NP-hard [7]. Can be approximated within factor 1/2 [11].

   Exercise 1: Show that local search (iteratively move to the other side a vertex if more than half of its neighbors are in the same side, while possible) yields 1/2-approximation.

   Exercise 2: Show that by randomly assigning vertices to either S or $\bar{S}$ the expected number of cut-edges is at least $|E|/2$.

2. **Quadratic Integer Program:**

$$\text{Max} \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$
$$\text{s.t. } x_i \in \{+1, -1\}, \quad \forall i \in V$$

   Relaxing the variables to be in $[-1, 1]$ does not give a linear program. Replacing $x_S \in \{0, 1\}$ with $x_S \geq 0$.

3. **Semidefinite programming relaxation:**

   A relaxation to a *vector program* can be obtained by assuming x_i is a unit-length vector in Euclidean space of large dimension m (instead of one dimension):

$$\text{Max} \sum_{(i,j)\in E} \frac{1 - v_i \cdot v_j}{2}$$
$$\text{s.t. } \|v_i\|_2 = 1, \quad \forall i \in V$$

   The above vector program is equivalent to the following semidefinite program by letting $y_{ij} = v_i \cdot v_j$:

$$\text{Max} \sum_{(i,j)\in E} \frac{1 - y_{ij}}{2}$$

$$\text{s.t. } y_{ii} = 1, \quad \forall i \in V$$

$$Y = (y_{ij}) \text{ is symmetric positive semidefinite.}$$

4. **Relaxation provides an upper bound:**

   Lemma 1: The SDP above can be solved in polynomial time within any desired accuracy.

   Lemma 2: SDP $\geq$ OPT.

   Importance of upper bound: Proving ALG $\geq \rho \cdot$ SDP will imply ALG $\geq \rho \cdot$ OPT.

   Example: For a 3-cycle, OPT = 2 while SDP = 9/4 by 3 vectors in the plane 120 degrees apart of each other.

5. **Hyperplane-cut rounding [4]:**

   Algorithm: Let $\{v_i\}$ be an optimal SDP solution in R^m. Choosen at random a vector r from the unit sphere S^m, and set $x_i = \text{sgn}(r \cdot v_i)$, i.e. $S = \{i \in V : r \cdot v_i \geq 0\}$.

   Geometric view: Choose a random hyperplane going through the origin (whose normal is r). It partitions the vectors (vertices) into two sides, forming a partition of V.

   Observations:

   (1) The rounding is invariant to rotation (just like the vector program).

   (2) Choosing a random vector from S^m can be done by choosing m iid Gaussians $X_1, \ldots, X_m$ and letting r be a unit-length vector in the direction $(X_1, \ldots, X_m)$. In fact, the same holds wrt to any orthogonal basis of R^m.

   Theorem 3: The cut producted by this algorithm has expected size at least $0.878 \cdot$ SDP.

6. **Claim:** For every $i, j \in V$, Pr[exactly one of i,j falls into S] $= \alpha_{ij}/\pi$, where $\alpha_{ij} \in [0, \pi]$ is the angle between v_i and v_j.

   Proof of claim: By the rotation invariance of r and of the SDP solution, we may assume that v_i and v_j are nonzero in all but the first two coordinates. Consequently, v_i and v_j lie in a two-dimensional plane, and for the event we are interested in, we may assume that $X_3 = \ldots = X_m = 0$, i.e. r is chosen *uniformly* from the unit circle in that plane. Using a two-dimensional picture, it is easy to verify that the probability the normal to r separates v_i from v_j is exactly α_{ij}/π.

7. **Proof of Theorem:**

   By the claim, for every $i, j \in V$, $\mathbb{E}[\frac{1-x_i x_j}{2}] = \alpha_{ij}/\pi$. By elementary calculus, the RHS is at least $0.878 \cdot (1 - \frac{\cos \alpha_{ij}}{2}) = 0.878 \frac{1 - v_i \cdot v_j}{2}$.

   Summing over all edges, we have by linearity of expectation, $\mathbb{E}[\text{ALG}] \geq 0.878 \cdot \text{SDP}$.

   Exercise 3: Suppose that SDP $= c|E|$ for some $1/2 < c < 1$. Show there exist c in this range, for which this rounding achieves a better approximation factor.

8. **Comments:**

1. The above rounding can be derandomized.

2. One can add additional constraints like the triangle inequality:

$$(v_i - v_k)^2 \leq (v_i - v_j)^2 + (v_j - v_k)^2, \quad \forall i, j, k \in V,$$

but they did not lead to an improved approximation factor for Max-Cut.

3. The integrality ratio of the SDP above is exactly what the randomized rounding gives (even with triangle inequality), i.e. $\rho_{\text{GW}} = \min_{\alpha \in [0,\pi]} \frac{\alpha/\pi}{(1-\cos\alpha)/2} \approx 0.878$. A 5-cycle gives a bound slightly worse than 0.878, but an exact bound requires considerable more work, see Delorme-Poljak [1, 2], Feige-Schechtman [3] and Khot-Vishnoi [9].

4. If the Unique Games conjecture is true, than it is NP-hard to achieve approximation factor better than $\rho_{\text{GW}} \approx 0.878$ [8, 10]. Otherweise, the hardness of approximation factor currently known is a bigger (worse) constant [12, 5].

5. A similar rounding procedure works for other problems like Max-DICUT and MAX-2SAT.

Two main differences: (1) There is an additional vector v_0 used to "distinguish" the two sides. (2) The triangle inequalities are useful to improve the approximation ratio.

6. The SDP rounding above motivated a more involved SDP rounding procedure for coloring 3-colorable graph [6].

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