References on Higher Laplacian Eigenvalues and Combinatorial Graph Properties

Luca Trevisan*

Last Revised September 19, 2014

This is a collection of references for a series of lectures that I gave in the "boot camp" of the semester on spectral graph theory at the Simons Institute in August, 2014. To keep this document focused, I only refer to results related to the lectures in question, which means that my own work is disproportionally represented. I plan to post a better version of this document in the future, so please send me corrections and additions.

1 Definitions

In the following, $G = (V, E)$ is an undirected graph, and $L = I - D^{-1/2}LD^{-1/2}$ is the normalized Laplacian of G, where D is the diagonal matrix of degrees; we denote by $d_v = D_{v,v}$ the degree of vertex v.

The Rayleigh quotient of a vector $x \in \mathbb{R}^V$ is

$$R(x) := \frac{\sum_{\{u,v\}\in E} |x_u - x_v|^2}{\sum_v d_v x_v^2}$$

(This is actually the Rayleigh quotient of $x\sqrt{D}$; hopefully this abuse of notation will not cause confusion.)

We denote the eigenvalues of L by the non-increasing sequence

$$0 = \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$$

*luca@berkeley.edu. Compute Science Division and Simons Institute for the Theory of Computing, U.C. Berkeley. This material is based upon work supported by the National Science Foundation under Grant No. 1216642 and by the US-Israel BSF Grant No. 2010451. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

The volume $vol(S)$ of a subset of vertices $S \subseteq V$ is the sum of the degrees, $vol(S) := \sum_v d_v$. The conductance of a nonempty set S is

$$\phi_G(S) := \frac{E(S, V-S)}{vol(S)}$$

note that

$$\phi_G(S) = R(\mathbf{1}_S)$$

We will usually omit the subscript. The conductance of a graph is

$$\phi(G) = \min_{S:\ vol(S) \leq \frac{1}{2} vol(S)} \phi_G(S)$$

A closely related notion is the (degree-weighted, normalized) uniform sparsity of a cut. The uniform sparsity of a set S is

$$\mathrm{us}_G(S) := vol(V) \cdot \frac{E(S, V-S)}{vol(S) \cdot vol(V-S)}$$

and the (degree-weighted) uniform sparsest cut problem on a graph is the optimization problem

$$\mathrm{usc}(G) = \min_S \ \mathrm{us}_G(S)$$

We have $\frac{1}{2}\mathrm{usc}(G) \leq \phi(G) \leq \mathrm{usc}(G)$, and there is a standard reduction from the problem of computing $\phi(G)$ in general graphs to computing $\phi(G)$ in bounded-degree regular graphs, with a constant loss in approximation, so provided that one is allowed to add constant terms to the approximation analysis, one may use uniform sparsest cut and conductance interchangeably, and one may assume regularity and bounded degree without loss of generality.

It is a classical result (I am not aware of what is a correct early reference) that $\lambda_k = 0$ if and only if G has at least k connected components. We discuss robust versions of this result and their applications.

The $k = 2$ case is well understood via the discrete Cheeger inequalities

$$\frac{\lambda_2}{2} \leq \phi(G) \leq \sqrt{2\lambda_2} \tag{1}$$

Dodziuk [Dod84] proves the discrete Cheeger inequality in the above form. Independently, a relation between *vertex* expansion and λ_2 was proved by Alon and Milman [Alo86, AM85]; following the argument of Alon and Milman, Sinclair and Jerrum [SJ89] and Lawler and Sokal [LS88] prove the above relation between conductance and second eigenvalue of the normalized Laplacian (in their setting, one minus the second eigenvalue of the random walk on the graph, which is the same). The work of Dodziuk and of Alon and Milman follows Cheeger's proof of his inequality concerning Riemann manifolds [Che70].

Importantly, the proof that $\phi(G) \leq \sqrt{2\lambda_2}$ is constructive, in the sense that given an eigenvector of λ_2 one finds a cut of expansion at most $\sqrt{2\lambda_2}$ in nearly linear time. The algorithm to find such a cut was first described by Fiedler [Fie73] and it has been widely used in practice. The work of Dodziuk was motivated by the goal of studying discrete analogs of results in Riemann geometry, the work of Alon and Milman was motivated by the goal of finding necessary and sufficient conditions for graph expansion, and the work of Sinclair and Jerrum and of Lawler and Sokal was motivated by the goal of understanding the mixing time of random walks. I am not sure at what point it was realized that the discrete Cheeger inequalities could be seen as a worst-case analysis of the quality of the cut produced by Fiedler's algorithm; the 1996 Spielman-Teng paper on planar graphs [ST96] takes it as a long established fact and refers to [AM85, SJ89].

Some of the results that we discuss below have only been proved in the case of regular graphs, but generally speaking all the techniques being used have natural generalizations to the case of irregular graphs.

2 Graphs in which λ_k is large

The theme of several results concerning graphs with large λ_k is that they are *easy instances* for approximation algorithms.

This was first realized by Kolla [Kol10], who proved that if one has an instance of Unique Games in which the associated label-extended graph has a sufficiently large value of λ_k then, under an additional assumption on the smoothness of the corresponding eigenvector, one can find a good approximation in time polynomial in the input size and exponential in k via the technique of *subspace enumeration*.

Arora, Barak and Steurer [ABS10] give a simpler proof of this result, without the smoothness requirement; combined with their "higher order Cheeger inequality" (see below), this gives their sub-exponential algorithm for small-set expansion and unique games.[1]

Concerning the approximation of conductance, Arora, Barak and Steurer show that one can get a $O(1/\lambda_k)$-factor approximation in time $exp(O(k)) \cdot n^{O(1)}$.

Barak, Raghavendra and Steurer [BRS11] show that one can use semidefinite pro-

[1]The paper of Arora, Barak, and Steurer, establishing properties of graphs in which λ_k is small, as well as properties of graphs in which λ_k is large, was the catalyst for much of the work on higher eigenvalues described in this document.

gramming to replicate the trade-off between running time and spectral assumption in the Arora-Barak-Steurer algorithms.

Guruswami and Sinop [GS13] show that the subspace enumeration algorithm of Arora, Barak and Steurer can be combined with an algorithm of Andersen and Lang [AL08] to show that if λ_k is a sufficiently large constant times $\phi(G)$, then one can find a constant factor approximation of $\phi(G)$ in time $exp(O(k)) \cdot n^{O(1)}$. For example, if $\lambda_k > 16\mathrm{usc}(G)$ then one finds 4-approximation of $\mathrm{usc}(G)$. Using hierarchies of semidefinite programs, Guruswami and Sinop prove that there is a constant c such that if $\lambda_k > c\epsilon^{-1}\mathrm{usc}(G)$ then a $(1+\epsilon)$-approximation of usc can be found in time $exp(O(k/\epsilon^2)) \cdot n^{O(1)}$.

Kwok et al. [KLL⁺13] prove that there is a constant c such that, for every k, if S_F is the cut found by Fiedler's algorithm using an eigenvector of λ_2, one has the refined Cheeger inequality

$$\phi(G) \leq \phi(S_F) \leq c \cdot k \cdot \frac{\lambda_2}{\sqrt{\lambda_k}} \tag{2}$$

which means that, in graphs in which λ_k is large for small k, one has an improved analysis of Fiedler's algorithm compared to what comes out of the standard Cheeger inequality.

Oveis-Gharan and Trevisan [OT13a] show that the ARV relaxation of conductance [ARV04] can be rounded with an approximation ratio of $O(\sqrt{\log k})$ provided that $\lambda_k \geq c \cdot (\log k)^{2.5} \cdot \phi(G)$, where c is an absolute constant.

Oveis-Gharan and Trevisan [OT13b] show that graphs in which λ_k is large for small k satisfy a *weak regularity Lemma* in the sense of Frieze and Kannan [FK96], and this, together with the Frieze-Kannan algorithms, recovers weaker versions of the approximation algorithms in [BRS11, GS13].

Arora, Ge and Sinop [AGS13] show that for every ϵ there is a c_ϵ such that one get a $(1+\epsilon)$ approximation to usc_G in time polynomial in n and exponential in k, provided that $\mathrm{SSE}_{\frac{1}{k}}(G) > c_\epsilon\sqrt{\log k}\sqrt{\log n} \cdot \mathrm{usc}_G$. Oveis-Gharan and Trevisan [OT13a] prove that constant-factor approximation is possible in the same running time provided that $\phi_k(G) > c \cdot \sqrt{\log k} \cdot \sqrt{\log n} \cdot \log\log n \cdot \mathrm{usc}_G$.

The algorithms in [BRS11, GS13, KLL⁺13, AGS13, OT13a] rely on the fact that, in graphs in which λ_k is large, certain convex relaxations have "structured" optimal and near-optimal solutions, which are easier to round than general solutions. The algorithms in [ABS10, OT13b] rely on the fact that near-optimal *combinatorial* solutions have a special form, and one can do complete enumeration over solutions having that form. It is not clear if there is a unified way of thinking about such algorithms.

A property of graphs in which λ_k is large is that the vertex set can be covered by sets of vertices each inducing an expander. In particular, Oveis-Gharan and Trevisan [OT14] prove that, in every graph, there exists a partition of the vertices into $\ell \leq k$ sets $(S_1 \ldots, S_\ell)$ such that, if we call G_i the subgraph induced by the vertex set S_i, we have $\phi_{G_i} \geq \Omega(\lambda_k/k^2)$.

Meka, Moitra and Srivastava [MMS] prove that one can cover a $1-\epsilon$ fraction of the vertices with $t = O_\epsilon(k)$ disjoint sets $S_1, \ldots, S_t$, such that each set S_i induces a subgraph G_i such that $\phi(G_i) \geq \Omega_\epsilon(\lambda_k/\log k)$.

3 Graphs in which λ_k is small

If $\lambda_k = 0$, then the graph has at least k connected components, that is, there is a partition of the vertices into k subsets such that each of them has conductance zero. Note also that one of them has size at most n/k. The following definitions allow us to formalize robust versions of this fact.

Define the order-k conductance of a collection of disjoint sets $S_1, \ldots, S_k$ as

$$\phi_k(S_1, \ldots, S_k) := \max_i \phi(S_i)$$

and the order-k conductance of a graph as

$$\phi_k(G) = \min_{S_1,\ldots,S_k \text{ disjoint, nonempty}} \phi_k(S_1, \ldots, S_k)$$

we can also define the oder-k conductance for partitions of a graph as

$$\phi_k^p(G) = \min_{S_1,\ldots,S_k \text{ partition of } V} \phi_k(S_1, \ldots, S_k)$$

Define also the small-set conductance at density δ of a graph as

$$\mathrm{SSE}_\delta(G) = \min_{S \subseteq V, vol(S) \leq \delta vol(V)} \phi(S)$$

Clearly one has

$$\mathrm{SSE}_{\frac{1}{k}}(G) = \phi_k(G)$$

Note that for $k = 2$ one has $\phi_2(G) = \phi_2^p(G) = \phi(G)$. In general one has

$$\phi_k(G) \leq \phi_k^p(G) \leq O(k) \cdot \phi_k(G)$$

and

$$\phi_k^p(G) \leq O(\epsilon^{-1}) \cdot \phi_{k\cdot(1+\epsilon)}(G)$$

Arora, Barak and Steurer [ABS10] prove

$$\mathrm{SSE}_{\frac{1}{k}\cdot n^{\gamma}} \leq \sqrt{\frac{\lambda_k}{\gamma^{O(1)}}} \tag{3}$$

Miclo [Mic08] defines the quantity ϕ_k^p and observes

$$\frac{\lambda_k}{2} \leq \phi_k^p$$

Lee, Oveis-Gharan and Trevisan [LOT12] prove

$$\phi_k \leq O(k^2) \cdot \lambda_k$$

which gives the higher-order inequalities conjectured by Miclo

$$\frac{\lambda_k}{2} \leq \phi_k \leq O(k^2) \cdot \sqrt{\lambda_k}$$

and

$$\frac{\lambda_k}{2} \leq \phi_k^p \leq O(k^3) \cdot \sqrt{\lambda_k}$$

Louis et al. [LRTV12] and Oveis-Gharan, Lee and Trevisan [LOT12] prove that, for a constant $c > 1$, we have

$$\phi_k \leq O(\sqrt{\log k}) \cdot \sqrt{\lambda_{ck}}$$

Up to the choice of the constant c, the above upper bound applies also to ϕ_k^p.

Lee, Oveis-Gharan and Trevisan [LOT12] prove that one can find k disjointly supported vectors such that each of them has Rayleigh quotient $k^{O(1)} \cdot \lambda_k$. It would be interesting to show that one can find, say, $k/2$ disjointly supported vectors such that each of them has Rayleigh quotient $O(\sqrt{\log n} \cdot \lambda_k)$.

It would be interesting to have an approximation of ϕ_k up to a factor dependent only on a function of the size of the input and of k. Louis and Makarychev [LM14] provide a bi-criteria approximation as follows: they are able to find disjoint sets $S_1, \ldots, S_k$ (or, equivalently, a partition) such that $\phi_k(S_1, \ldots, S_k) \leq O(\sqrt{\log k}\sqrt{\log n}) \cdot \phi_{ck}$, for a constant $c > 1$.

4 Graphs in which λ_k is small and λ_{k+1} is large

Tanaka [Tan12] proves that if $\phi_{k+1} > 3^{k+1}\phi_k$, then there is a partition of V into k sets $S_1, \ldots, S_k$ such that each set has small conductance, but also each set induces a subgraph of large expansion, that is, we have

$$\forall i \in \{1, \ldots, k\}: \quad \phi_{G_i} \geq 3^{-(k+1)}\phi_{k+1}, \quad \phi(S_i) \leq 3^k \phi_k$$

where G_i is the vertex-induced subgraph of G induced by S_i. The result is non-algorithmic. Because of the higher-order Cheeger inequalities mentioned above, a sufficient condition for $\phi_{k+1} > 3^{k+1}\phi_k$ is to have $\lambda_{k+1} > c \cdot k^3 \cdot 3^{k+1}\sqrt{\lambda_k}$, for a certain constant c.

Kannan, Vempala, Vetta [KVV04] argue that a partition of the vertices into a family of subsets such that each set has small conductance in the graph but induces a subgraph of large conductance is a good clustering of the graph.

Oveis-Gharan and Trevisan [OT14] show that if $\phi_{k+1} > (1+\epsilon)\phi_k$, then there is a partition of the vertices into k subsets $(S_1, \ldots, S_k$ such that

$$\forall i \in \{1, \ldots, k\} : \quad \phi_{G_i} \geq \Omega\left(\frac{\epsilon}{k}\right) \cdot \phi_{k+1}, \quad \phi(S_i) \leq k\phi_k$$

5 Graphs in which λ_n is large

Recall that we have

$$\lambda_n = \max_{x \in \mathbb{R}^n} \frac{\sum_{\{u,v\} \in E} |x_u - x_v|^2}{\sum_v d_v x_v^2} = 2 - \min_{x \in \mathbb{R}^n} \frac{\sum_{\{u,v\} \in E} |x_u + x_v|^2}{\sum_v d_v x_v^2}$$

Graphs in which $\lambda_n = 2$ are such that there is a connected component (possibly, the entire graph) which is bipartite. We define a combinatorial quantity which is zero when the graph has a bipartite connected component, and which, in general, measures the "distance" from such a situation. Following [Tre09], we call it the bipartiteness ratio $\beta(G)$.

For a vector $z \in \{-1, 0, 1\}$, we define

$$\beta(z) = \frac{1}{2} \frac{\sum_{\{u,v\} \in E} |z_u + z_v|}{\sum_v d_v |z_v|}$$

A more combinatorial view of the above function is that we can view z as defining a subset $S \subseteq V$ and a bipartition $(B, S - B)$ of S, and the parameter β counts the number of edges in the subgraph induces by S that are not cut by the partition, and adds half of the edges leaving S; the quantity is normalized by dividing by the volume of S.

We have the Cheeger-like inequalities [Tre09] (also later rediscovered by Bauer and Jost [BJ13]):

$$\frac{2 - \lambda_2}{2} \leq \beta(z) \leq \sqrt{2 \cdot (2 - \lambda_n)}$$

Liu [Liu14] formulates a higher order Cheeger inequality for λ_{n-k} analogous to the theory described above for λ_k.

6 Hypergraphs?

It would be excellent to have an algorithmic "spectral" theory for hyper-graphs. A bottleneck for such a theory is that one of the problems that are addressed by spectral graph theory is conjectured to be intractable for hypergraphs: given a sparse random graph, one can efficiently certify that it is "quasirandom" in the sense that it satisfies the expander mixing lemma via spectral. Certifying quasirandomness of sparse hypergraphs, however, would contradict Feige's conjecture on the intractability of certifying unstatisfiability of sparse random instances of 3SAT.[2]

Friedman and Wigderson [FW95] define a non-algorithmic spectral theory of hypergraphs. Louis [Lou14] describes an algorithmically treatable spectral theory of hypergraphs, which includes an analog of Cheeger's inequality.

References

[ABS10] Sanjeev Arora, Boaz Barak, and David Steurer. Subexponential algorithms for unique games and related problems. In *Proceedings of the 51st IEEE Symposium on Foundations of Computer Science*, 2010. 3, 4, 6

[AGS13] Sanjeev Arora, Rong Ge, and Ali Kemal Sinop. Towards a better approximation for sparsest cut? In *Proceedings of the 54th IEEE Symposium on Foundations of Computer Science*, pages 270–279, 2013. 4

[AL08] Reid Andersen and Kevin. J. Lang. An algorithm for improving graph partitions. In *Proceedings of the 19th ACM-SIAM Symposium on Discrete Algorithms*, page 651?660, 2008. 4

[Alo86] Noga Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986. 2

[AM85] N. Alon and V.D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *Journal of Combinatorial Theory, Series B*, 38(1):73–88, 1985. 2, 3

[2]See http://terrytao.wordpress.com/2008/02/15/luca-trevisan-checking-the-quasirandomness-of-graphs-and-hypergraphs/ for a discussion of this issue.

[ARV04] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows and a $\sqrt{\log n}$-approximation to sparsest cut. In *Proceedings of the 36th ACM Symposium on Theory of Computing*, 2004. 4

[BJ13] F. Bauer and J. Jost. Bipartite and neighborhood graphs and the spectrum of the normalized graph Laplacian. *Comm. Anal. Geom.*, 21(4):787–845, 2013. 7

[BRS11] Boaz Barak, Prasad Raghavendra, and David Steurer. Rounding semidefinite programming hierarchies via global correlation. In *Proceedings of the 52nd IEEE Symposium on Foundations of Computer Science*, pages 472–481, 2011. 3, 4

[Che70] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. Princeton Univ. Press, 1970. 3

[Dod84] Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984. 2

[Fie73] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973. 3

[FK96] Alan Frieze and Ravi Kannan. The regularity lemma and approximation schemes for dense problems. In *Proceedings of the 37th IEEE Symposium on Foundations of Computer Science*, pages 12–20, 1996. 4

[FW95] Joel Friedman and Avi Wigderson. On the second eigenvalue of hypergraphs. *Combinatorica*, 15(1):43–65, 1995. 8

[GS13] Venkatesan Guruswami and Ali Kemal Sinop. Approximating non-uniform sparsest cut via generalized spectra. In *Proceedings of the 24th ACM-SIAM Symposium on Discrete Algorithms*, pages 295–305, 2013. 4

[KLL+13] Tsz Chiu Kwok, Lap Chi Lau, Yin Tat Lee, Shayan Oveis Gharan, and Luca Trevisan. Improved Cheeger's inequality: analysis of spectral partitioning algorithms through higher order spectral gap. In *Proceedings of the 45th ACM Symposium on Theory of Computing*, pages 11–20, 2013. 4

[Kol10] Alexandra Kolla. Spectral algorithms for unique games. In *Proceedings of the 25th IEEE Conference on Computational Complexity*, pages 122–130, 2010. 3

[KVV04] Ravi Kannan, Santosh Vempala, and Adrian Vetta. On clusterings: Good, bad and spectral. *Journal of the ACM*, 51(3):497–515, 2004. 7

[Liu14] Shiping Liu. Multi-way dual cheeger constants and spectral bounds of graphs. *arXiv preprint arXiv:1401.3147*, 2014. 8

[LM14] Anand Louis and Konstantin Makarychev. Approximation algorithm for sparsest k-partitioning. In *Proceedings of the 25th ACM-SIAM Symposium on Discrete Algorithms*, pages 1244–1255, 2014. 6

[LOT12] James R. Lee, Shayan Oveis Gharan, and Luca Trevisan. Multi-way spectral partitioning and higher-order Cheeger inequalities. In *Proceedings of the 44th ACM Symposium on Theory of Computing*, pages 1117–1130, 2012. 6

[Lou14] Anand Louis. Hypergraph markov operators, eigenvalues and approximation algorithms. Technical report, 2014. arXiv:1408.2425. 8

[LRTV12] Anand Louis, Prasad Raghavendra, Prasad Tetali, and Santosh Vempala. Many sparse cuts via higher eigenvalues. In *Proceedings of the 44th ACM Symposium on Theory of Computing*, pages 1131–1140, 2012. 6

[LS88] Gregory F. Lawler and Alan D. Sokal. Bounds on the L2 spectrum for Markov chains and Markov processes: a generalization of Cheeger's inequality. *Trans. Amer. Math. Soc.*, 309(2):557?580, 1988. 2

[Mic08] Laurent Miclo. On eigenfunctions of Markov processes on trees. *Probability Theory and Related Fields*, 142(3-4):561–594, 2008. 6

[MMS] Raghu Meka, Ankur Moitra, and Nikhil Srivastava. Personal communication. 5

[OT13a] Shayan Oveis Gharan and Luca Trevisan. Improved ARV rounding in small-set expanders and graphs of bounded threshold rank. Technical report, 2013. arXiv:1304.2060. 4

[OT13b] Shayan Oveis Gharan and Luca Trevisan. A new regularity lemma and faster approximation algorithms for low threshold rank graphs. In *APPROX-RANDOM*, pages 303–316, 2013. 4

[OT14] Shayan Oveis Gharan and Luca Trevisan. Partitioning into expanders. In *Proceedings of the 25th ACM-SIAM Symposium on Discrete Algorithms*, 2014. 4, 7

[SJ89] Alistair Sinclair and Mark Jerrum. Approximate counting, uniform generation and rapidly mixing Markov chains. *Information and Computation*, 82(1):93–133, 1989. 2, 3

[ST96] Daniel A. Spielman and Shang-Hua Teng. Spectral partitioning works: Planar graphs and finite element meshes. In *Proceedings of the 37th IEEE Symposium on Foundations of Computer Science*, pages 96–105, 1996. 3

[Tan12] Mamoru Tanaka. Higher eigenvalues and partitions of a graph. Technical report, 2012. arXiv:1112.3434. 6

[Tre09] Luca Trevisan. Max Cut and the smallest eigenvalue. In *Proceedings of the 41st ACM Symposium on Theory of Computing*, pages 263–272, 2009. 7