Lecture 17

In which we define and analyze the zig-zag graph product.

1 Replacement Product and Zig-Zag Product

In the previous lecture, we claimed it is possible to "combine" a d-regular graph on D vertices and a D-regular graph on N vertices to obtain a d^2-regular graph on ND vertices which is a good expander if the two starting graphs are. Let the two starting graphs be denoted by H and G respectively. Then, the resulting graph, called the *zig-zag product* of the two graphs is denoted by $G \textcircled{z} H$.

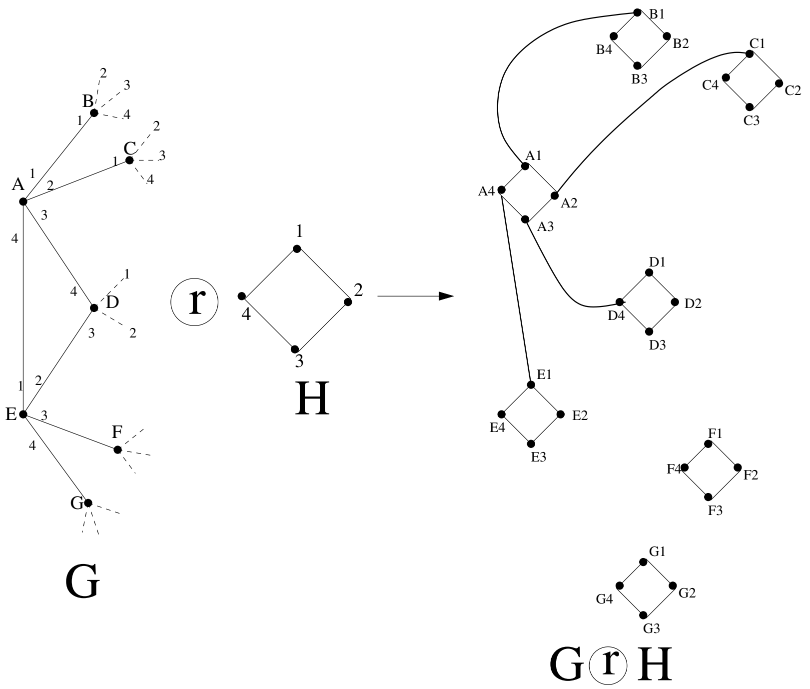
Using $\lambda(G)$ to denote the eigenvalue with the second-largest absolute value for a graph G, we claimed that if $\lambda(H) \leq b$ and $\lambda(G) \leq a$, then $\lambda(G \textcircled{z} H) \leq a + 2b + b^2$. In this lecture we shall describe the construction for the zig-zag product and prove this claim.

2 Replacement product of two graphs

We first describe a simpler product for a "small" d-regular graph on D vertices (denoted by H) and a "large" D-regular graph on N vertices (denoted by G). Assume that for each vertex of G, there is some ordering on its D neighbors. Then we construct the replacement product (see figure) $G \textcircled{r} H$ as follows:

- Replace each vertex of G with a copy of H (henceforth called a *cloud*). For $v \in V(G), i \in V(H)$, let (v, i) denote the i^{th} vertex in the v^{th} cloud.
- Let $(u, v) \in E(G)$ be such that v is the i-th neighbor of u and u is the j-th neighbor of v. Then $((u, i), (v, j)) \in E(G \textcircled{r} H)$. Also, if $(i, j) \in E(H)$, then $\forall u \in V(G)$ $((u, i), (u, j)) \in E(G \textcircled{r} H)$.

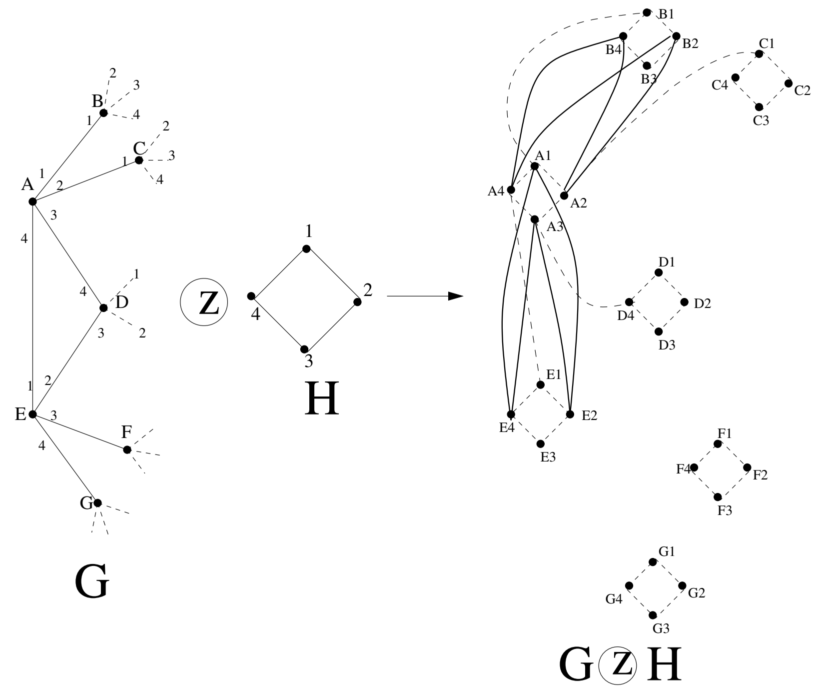
Note that the replacement product constructed as above has ND vertices and is $(d + 1)$-regular.



3 Zig-zag product of two graphs

Given two graphs G and H as above, the zig-zag product $G\textcircled{z}H$ is constructed as follows (see figure):

- The vertex set $V(G\textcircled{z}H)$ is the same as in the case of the replacement product.
- $((u,i),(v,j)) \in E(G\textcircled{z}H)$ if there exist ℓ and k such that $((u,i)(u,\ell)$, $((u,\ell),(v,k))$ and $((v,k),(v,j))$ are in $E(G\textcircled{r}H)$ i.e. (v,j) can be reached from (u,i) by taking a step in the first cloud, then a step between the clouds and then a step in the second cloud (hence the name!).



It is easy to see that the zig-zag product is a d^2-regular graph on ND vertices.

Let $M \in \mathbb{R}^{([N]\times[D])\times([N]\times[D])}$ be the normalized adjacency matrix of $G \textcircled{z} H$. Using the fact that each edge in $G \textcircled{r} H$ is made up of three steps in $G \textcircled{r} H$, we can write M as BAB, where

$$B[(u,i),(v,j)] = \begin{cases} 0 & if\ u \neq v \\ M_H[i,j] & if\ u = v \end{cases}$$

And $A[(u,i),(v,j)] = 1$ if u is the j-th neighbor of v and v is the i-th neighbor of u, and $A[(u,i),(v,j)] = 0$ otherwise.

Note that A is the adjacency matrix for a matching and is hence a permutation matrix.

4 Preliminaries on Matrix Norms

Recall that, instead of bounding λ_2, we will bound the following parameter (thus proving a stronger result).

Definition 1 *Let M be the normalized adjacency matrix of a graph $G = (V,E)$, and $\lambda_1 \geq \ldots \geq \lambda_n$ be its eigenvalues with multiplicities. Then we use the notation*

$$\lambda(M) := \max_{i=2,\ldots,n} \{|\lambda_i|\} = \max\{\lambda_2, -\lambda_n\}$$

The parameter λ has the following equivalent characterizations.

Fact 2

$$\lambda(M) = \max_{\mathbf{x} \in \mathbb{R}^V - \{\mathbf{0}\}, \mathbf{x} \perp \mathbf{1}} \frac{||Mx||}{||x||} = \max_{\mathbf{x} \in \mathbb{R}^v, \mathbf{x} \perp \mathbf{1}, ||\mathbf{x}||=1} ||Mx||$$

Another equivalent characterization, which will be useful in several contexts, can be given using the following matrix norm.

Definition 3 (Spectral Norm) *The spectral norm of a matrix $M \in \mathbb{R}^{n \times n}$ is defined as*

$$||M|| = \max_{\mathbf{x} \in \mathbb{R}^V, ||\mathbf{x}||=1} ||Mx||$$

If M is symmetric with eigenvalues $\lambda_1, \ldots, \lambda_n$, then the spectral norm is $\max_i |\lambda_i|$. Note that M is indeed a norm, that is, for every two square real matrices A, B we have $||A + B|| \leq ||A|| + ||B||$ and for every matrix A and scalar α we have $||\alpha A|| = \alpha ||A||$. In addition, it has the following useful property:

Fact 4 *For every two matrices $A, B \in \mathbb{R}^{n \times n}$ we have*

$$||AB|| \leq ||A|| \cdot ||B||$$

Proof: For every vector x we have

$$||AB\mathbf{x}|| \leq ||A|| \cdot ||B\mathbf{x}|| \leq ||A|| \cdot ||B|| \cdot ||\mathbf{x}||$$

where the first inequality is due to the fact that $||Az|| \leq ||A|| \cdot ||\mathbf{z}||$ for every vector $\mathbf{z}$, and the second inequality is due to the fact that $||B\mathbf{x}|| \leq ||B|| \cdot ||\mathbf{x}||$. So we have

$$\min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq \mathbf{0}}$$

$\square$

We can use the spectral norm to provide another characterization of the parameter $\lambda(M)$ of the normalized adjacency matrix of a graph.

Lemma 5 *Let G be a regular graph and $M \in \mathbb{R}^{n \times n}$ be its normalized adjacency matrix. Then*

$$\lambda(M) = ||M - \frac{1}{n} J||$$

where J is the matrix with a 1 in each entry.

Proof: Let $\lambda_1 = 1 \geq \lambda_2 \geq \cdots \lambda_n$ be the eigenvalues of M and $\mathbf{v}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ a corresponding system of orthonormal eigenvector. Then we can write

$$M = \lambda_1 \mathbf{v}_1 \mathbf{v}_1^T + \cdots + \lambda_n \mathbf{v}_n \mathbf{v}_n^T$$

Noting that $\mathbf{v}_1 \mathbf{v}_1^T = \frac{1}{n}J$, we have

$$M - \frac{1}{n}J = 0 \cdot \mathbf{v}_1 \mathbf{v}_2^T + \sum_{i=2}^{n} \lambda_i \mathbf{v}_i \mathbf{v}_i^T$$

and so $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is also a system of eigenvectors for $M - \frac{1}{n}J$, with corresponding eigenvalues $0, \lambda_2, \ldots, \lambda_n$, meaning that

$$||M - \frac{1}{n}J|| = \max\{0, \lambda_2, \ldots, \lambda_n\} = \lambda(M)$$

□

The above lemma has several applications. It states that, according to a certain definition of distance, when a graph is a good expander then it is close to a clique. (The matrix $\frac{1}{n}J$ is the normalized adjacency matrix of a clique with self-loops.) The proof of several results about expanders is based on noticing that the result is trivial for cliques, and then on "approximating" the given expander by a clique using the above lemma.

We need one more definition before we can continue with the analysis of the zig-zag graph product.

Definition 6 (Tensor Product) *Let $A \in \mathbb{R}^{N \times N}$ and $B \in \mathbb{R}^{D \times D}$ be two matrices. Then $A \otimes B \in \mathbb{R}^{ND \times ND}$ is a matrix whose rows and columns are indexed by pairs $(u, i) \in [N] \times [D]$ such that*

$$(A \otimes B)_{(u,i),(v,j)} = A_{u,v} \cdot B_{i,j}$$

For example $I \otimes M$ is a block-diagonal matrix in which every block is a copy of M.

5 Analysis of the Zig-Zag Product

Suppose that G and H are identical cliques with self-loops, that is, are both n-regular graphs with self-loops. Then the zig-zag product of G and H is well-defined, because the degree of G is equal to the number of vertices of H. The resulting graph $G ⓩ H$ is a n^2-regular graph with n^2 vertices, and an inspection of the definitions reveals that $G ⓩ H$ is indeed a clique (with self-loops) with n^2 vertices.

The intuition for our analysis is that we want to show that the zig-zag graph product "preserves" distances measured in the matrix norm, and so if G is close (in matrix norm) to a clique and H is close to a clique, then $G \textcircled{z} H$ is close to the zig-zag product of two cliques, that is, to a clique. (Strictly speaking, what we just said does not make sense, because we cannot take the zig-zag product of the clique that G is close to and of the clique that H is close to, because they do not have the right degree and number of vertices. The proof, however, follows quite closely this intuition.)

Theorem 7 *If $\lambda(M_G) = a$ and $\lambda(M_H) = b$, then*

$$\lambda(G \textcircled{z} H) \leq a + 2b + b^2$$

PROOF: Let M be the normalized adjacency matrix of $G \textcircled{z} H$, and let $\mathbf{x}$ be a unit vector such that $\mathbf{x} \perp \mathbf{1}$ and

$$\lambda(M) = ||M\mathbf{x}||$$

Recall that we defined a decomposition

$$M = BAB$$

where A is a permutation matrix, and $B = I \otimes M_H$. Let us write $E := M_H - \frac{1}{D}J$, then $B = I \otimes \frac{1}{D}J + I \otimes E$. Let us call $\bar{J} := I \otimes \frac{1}{D}J$ and $\bar{E} := I \otimes E$.

First, we argue that the matrix norm of $\bar{E}$ is small. Take any vector $\mathbf{z} \in \mathbb{R}^{ND}$ and write is as $\mathbf{z} = (\mathbf{z}_1, \ldots, \mathbf{z}_N)$, where, for each $u \in [N]$, $\mathbf{z}_u$ is the D-dimensional restriction of $\mathbf{z}$ to the coordinates in the cloud of u. Then

$$||(I \otimes E)\mathbf{z}||^2 = \sum_u ||E\mathbf{z}_u||^2 \leq \sum_u ||E||^2 \cdot ||\mathbf{z}_u||^2 = ||E||^2 \cdot ||\mathbf{z}||^2$$

and so we have

$$||I \otimes E|| \leq ||E|| \leq b$$

Then we have

$$BAB = (\bar{J} + \bar{E})A(\bar{J} + \bar{E})$$
$$= \bar{J}A\bar{J} + \bar{J}A\bar{E} + \bar{E}A\bar{J} + \bar{E}A\bar{A}$$

and so, using the triangle inequality and the property of the matrix norm, we have

$$||BAB\mathbf{x}|| \leq ||\bar{J}A\bar{J}\mathbf{x}|| + ||\bar{E}A\bar{J}|| + ||\bar{J}A\bar{E}|| + ||\bar{E}A\bar{E}||$$

where

$$||\bar{E}A\bar{J}|| \leq ||\bar{E}|| \cdot ||A|| \cdot ||\bar{J}|| \leq ||\bar{E}|| \leq b$$

$$||\bar{J}A\bar{E}|| \leq ||\bar{J}|| \cdot ||A|| \cdot ||\bar{E}|| \leq ||\bar{E}|| \leq b$$
$$||\bar{E}A\bar{E}|| \leq ||\bar{E}|| \cdot ||A|| \cdot ||\bar{E}|| \leq ||\bar{E}||^2 \leq b^2$$

It remains to prove that $||\bar{J}A\bar{J}\mathbf{x}|| \leq a$. If we let $A_G = DM_G$ be the adjacency matrix of G, then we can see that

$$(\bar{J}A\bar{J})_{(u,i),(v,j)} = \frac{1}{D^2}(A_G)_{u,v} = \frac{1}{D}(M_G)_{u,v} = (M_G \otimes \frac{1}{D}J)_{(u,i),(v,j)}$$

That is,

$$\bar{J}A\bar{J} = M_G \otimes \frac{1}{D}J$$

Finally, we write $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, where $\mathbf{x}_u$ is the D-dimensional vector of entries corresponding to the cloud of u, we call $y_u := \sum_i \mathbf{x}_u(i)/D$, and we note that, by Cauchy-Schwarz:

$$||\mathbf{y}||^2 = \sum_u \left(\sum_i \frac{1}{D}\mathbf{x}_{u,i} \right)^2 \leq \sum_u \left(\sum_i \frac{1}{D}^2 \right) \cdot \left(\sum_i \mathbf{x}_{u,i}^2 \right) = \frac{1}{D}||\mathbf{x}||^2$$

The final calculation is:

$$\begin{aligned}
||\bar{J}A\bar{J}\mathbf{x}||^2 &= \left|| \left(M_G \otimes \frac{1}{D}J \right) \mathbf{x} \right||^2 \\
&= \sum_{u,i} \left(\sum_{v,j} \frac{1}{D}(M_G)_{u,v}\mathbf{x}_{u,i} \right)^2 \\
&= \sum_{u,i} \left(\sum_v (M_G)_{u,v} y_u \right)^2 \\
&= D \cdot \sum_u \left(\sum_v (M_G)_{u,v} y_u \right)^2 \\
&= D \cdot ||M_G \mathbf{y}||^2 \\
&\leq D \cdot a^2 \cdot ||\mathbf{y}||^2 \\
&\leq a^2 \cdot ||\mathbf{x}^2||^2
\end{aligned}$$

□