Notes for Lecture 23

Preliminary version

Notes on Reingold's Theorem, Part II

Today we continue the proof that the *undirected* (s,t)-connectivity problem can be solved in *deterministic* logarithmic space [Rei04]. We introduce the zig-zag graph product, an operation that can be used to reduce the degree of a graph without affecting much the eigenvalue gap. A sequence of zig-zag and powering operations, starting from an arbitrary connected graph, leads to a graph with a constant eigenvalue gap. The connectivity algorithm relies on such a construction.

1 The Zig-Zag Graph Product

Recall that if G is an undirected graph and A is its normalized adjacency matrix (its "random walk matrix"), and if we denote by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ the eigenvalues of A, in sorted order, then $\lambda_1 = 1$ and we use the notation

$$\lambda(G) := \max_{i=2,\ldots,n} |\lambda_i|$$

Furthermore, if G is d-regular, connected, and not bipartite, then $\lambda(G) \leq 1 - 1/dn^2$. Finally, we use the notation

$$\gamma(G) := 1 - \lambda(G)$$

and we call $\gamma(G)$ the *eigenvalue gap* of G. Recall that we proved that the diameter of G is at most $\log_{1/\lambda} n$, which is $O(\frac{1}{\gamma}\log n)$, where n is the number of vertices. In particular, if G is a graph with eigenvalue gap at least $1/2$, then the diameter is at most $\log_2 n$.

The problem of *constructing* graphs with a large eigenvalue gap is very difficult and it has been the subject of a lot of work. A recent construction of such graphs [RVW02] introduces a graph product called the *zig-zag*.

If G and H are graphs with large eigenvalue gap, the zig-zag product define a new graph $G \circledz H$ that has still a large eigenvalue gap and is bigger than G and H. The point is that one can use the product to start from very small graphs with large gap, that are easier to construct, and then "bootstrap" them to get bigger and bigger ones.

For the product to work, G must be a regular graph, say, with N vertices and degree D, and H must have D vertices, as many as the degree of G. If H is a d-regular graph, then $G \circledz H$ is a d^2-regular graph with ND vertices. (It may help to think of $N = 100,000$, $D = 1,000$ and $d = 10$ to get a sense of the parameters.)

In $G \circledz H$ there is a vertex $[v,a]$ for every vertex v of G and vertex a of H. There is an edge between $[v,a]$ and $[w,b]$ in $G \circledz H$ provided that there are vertices a', b' in H such that the edges (a, a') and (b', b) are in H, w is the a'-th neighbor of v in G, and v is the b'-th neighbor of w in G. (This makes more sense given the picture I drew in class.)

In [RVW02] there are various formulas relating $\gamma(G ⓩ H)$ to $\gamma(G)$ and $\lambda(H)$. If we restrict ourselves to special case $\gamma(H) \geq 1/2$, then we have the following theorem:

$$\text{If } \gamma(H) \geq \frac{1}{2}, \text{ then } \gamma(G ⓩ H) \geq \frac{3}{8}\gamma(G) \tag{1}$$

2 Combining Zig-Zag and Powering

Suppose now that H is a d-regular graph with d^{16} nodes (this choice will be clear later) and with $\gamma(H) \geq 1/2$. Let G be a d^{16}-regular graph. Consider the graph $G' := (G ⓩ H)^8$. (By this notation, we mean the graph whose adjacency matrix is the 8-th power of the adjacency matrix of $(G ⓩ H)$.) Then, we claim that

$$\gamma(G') \geq \min\left\{\frac{1}{2}\ ,\ 2\gamma(G)\right\} \tag{2}$$

and, furthermore, G' is a graph with Nd^{16} vertices and degree d^{16}. In other words, the new graph G' has a constant factor more vertices than G, the same degree as G, and an eigenvalue gap than is either twice that of G or at least $1/2$.

To prove that Equation 2 holds, first remember that if G is a graph, then $\lambda(G^k) = (\lambda(G))^k$ and so $\gamma(G^k) = 1 - (1 - \gamma(G))^k$. We prove Equation 2 by considering two cases. If $\gamma(G) \geq 1/4$, then $\gamma(G ⓩ H) \geq 3/32$ by Equation 1 and

$$\gamma((G ⓩ H)^8) = 1 - (1 - \gamma(G ⓩ H))^8 \geq 1 - \left(\frac{29}{32}\right)^8 = .545\ldots > \frac{1}{2}$$

If $\gamma(G) \leq 1/4$, then

$$\gamma((G ⓩ H)^8) = 1 - (1 - \gamma(G ⓩ H))^8 \geq 8 \cdot \frac{3}{8}\gamma(G) - 28\left(\frac{3}{8}\gamma(G)\right)^2 \geq 2\gamma(G)$$

where we used the fact that $(1 - \epsilon)^8 > 8\epsilon - \binom{8}{2}\epsilon^2$ for $0 < \epsilon < 1$. This completes the proof of Equation 2.

3 Reingold's Algorithm

As before, let H be a d-regular graph with d^{16} nodes and with $\gamma(H) \geq 1/2$. Such a graph exists for small values of d. In particular, $d = 8$ can be proved to be enough using the probabilistic method. An explicit construction is known for $d = 14$.

Let G be a d^{16}-regular graph G with n vertices that is connected and not bipartite. Consider the following recursively defined family of graphs

$$G_0 := G$$

$$G_i := (G_{i-1} ⓩ H)^8\ .$$

Then $\gamma(G_i) \geq \min\{1/2, \gamma(G) \cdot 2^i\}$. Since $\gamma(G) \geq 1/d^{16}n^2$, we have that for $t = \log_2 d^{16}n^2 = O(\log n)$ we have $\lambda(G_t) \geq 1/2$.

The number of vertices of G_t is $n \cdot \left(d^{16}\right)^t$, which is polynomial in n.

Finally, G_t can be constructed in log-space given G and t. More specifically, there is an algorithm that, given a graph G with n vertices and degree d^{16}, a graph H with d^{16} vertices and degree d, an integer t, and the names of two vertices v and w of G_t, uses $O(\log n + t)$ space and decides whether v and w are adjacent in G_t.

Suppose now that G is d^{16}-regular graph that is not necessarily connected, but such that each connected component is not bipartite. Then each connected component of G is a connected, non-bipartite, regular graph. If we apply the above construction to a connected component of G we obtain a graph with eigenvalue gap at least $1/2$. Indeed, if we apply the above construction to all of G, we can see that the result is the same as if we had applied the construction to each connected component of G separately.

In conclusion, if G is a d^{16}-regular graph with n vertices that has k connected components, each non-bipartite, then G_t, where $t = \log_2 d^{16}n^2$, is a d^{16}-regular graph with k connected components, and each connected component of G_t is a connected graph with eigenvalue gap at least $1/2$.

If s and v are vertices in G, then let s_t and v_t be vertices of G_t that "correspond" to s and v. Such vertices can be defined recursively, by letting $s_0 := s$, $v_0 = v$, and then letting s_i be one of the vertices of G_i in the block of vertices that replace s_{i-1} (similarly for v_i). By induction, one can see that s and v are connected in G if and only if s_t and v_t are connected in G_t. Finally, deciding whether s_t and v_t are connected in G_t is easy because we know that the diameter of each connected component of G_t is at most $\log_2 n \cdot d^{16t} = O(\log n)$, where $n \cdot d^{16t}$ is the number of vertices of G_t. Since each connected component of G_t has degree $d^{16} = O(1)$ and diameter $\log_2 n \cdot d^{16t} = O(\log n)$, it follows that one can explore the entire connected component of s_t, and verify whether v_t is an element of it, using $O(\log n)$ space.

What we have described so far gives an $O(\log n)$ space algorithm to decide (s,t)-connectivity in undirected graphs that are: (i) regular of degree d^{16} and (ii) such that each connected component is not bipartite.

It is easy to reduce the general case to this special case. First, one can reduce (s,t)-connectivity in general graph to (s,t)-connectivity in 3-regular graphs. This can be done by replacing each vertex v of G of degree d_v by a cycle of length d_v, and then placing the edges of G as a matching in the new graph. That is, if, in G, w is the i-th neighbor of v and v is the j-th neighbor of w, then we put an edge between the i-th vertex in the cycle of v and the j-th vertex in the cycle of w. Then we can add $d^{16} - 3$ self-loops to each vertex. This way the graph cannot be bipartite, and it has become regular of degree d^{16}.

References

[Rei04] Omer Reingold. Undirected st-connectivity in log-space. Technical Report TR04-94, Electronic Colloquium on Computational Complexity, 2004. 1

[RVW02] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Annals of Mathematics*, 155(1):157–187, 2002. 1, 2