# CS II Lecture 3

- FFT
- Graphs
  - Definitions, representations, connectivity

$f : \{0, 1, 2, \dots, n-1\} \to \mathbb{C}$

Linear space of dimension $n$

$$f(x) = \sum_{s=0}^{n-1} \hat{f}(s)\, g_s(x)$$

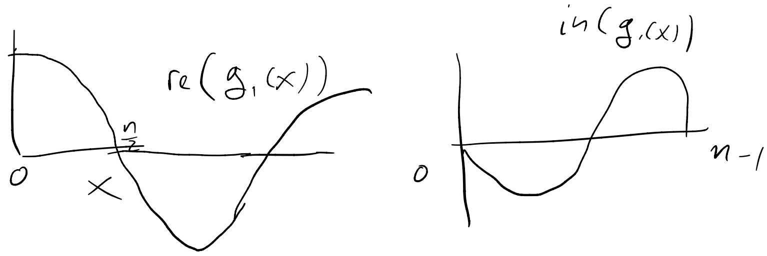
$s = 0, 1, \dots, n-1$

$$g_s(x) = e^{\frac{-2\pi i}{n} \cdot x \cdot s}$$

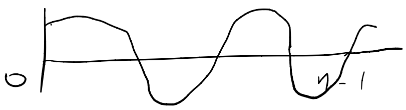
want to prove $g_s$ are linearly indep.

$$= \cos\left(\frac{2\pi}{n} \cdot x \cdot s\right) - i \sin\left(\frac{2\pi}{n} \cdot x \cdot s\right)$$

$$g_1(x) = \cos\left(2\pi \frac{x}{n}\right) - i \sin\left(2\pi \frac{x}{n}\right)$$




$g_2(x)$




$$\langle f, h \rangle \stackrel{def}{=} \sum_j f(j)\, \overline{h(j)}$$

Claim: functions $g_s$ are orthogonal to one another

Proof: choose $s \neq t$ $\quad s, t \in \{0, \dots, n-1\}$

$$\langle g_s, g_t \rangle = \sum_{j=0}^{n-1} \underbrace{e^{\frac{-2\pi i \cdot j \cdot s}{n}}}_{g_s(j)} \cdot \underbrace{e^{\frac{2\pi i \cdot j \cdot t}{n}}}_{\overline{g_t(j)}}$$

$$\langle g_s, g_t \rangle = \sum_{j=0}^{n-1} \underbrace{e^{\frac{-2\pi i \cdot j \cdot s}{n}}}_{g_s(j)} \cdot \underbrace{e^{\frac{2\pi i \cdot j \cdot t}{n}}}_{\overline{g_t}(j)}$$

$$= \sum_{j=0}^{n-1} e^{\frac{-2\pi i}{n} \cdot j \cdot (s-t)}$$

call $z = e^{\frac{-2\pi i}{n} \cdot (s-t)} \neq 1$ if $s \neq t$

$$= \sum_{j=0}^{n-1} z^j$$

note $z^n = e^{-2\pi i \cdot (s-t)} = 1$

$z^0 = 1$

$$= \sum_{j=1}^{n} z^j = z \cdot \sum_{j=0}^{n-1} z^j$$

$$\underbrace{(1-z)}_{\neq 0} \cdot \sum_{j=0}^{n-1} z^j = 0$$

$$\sum_{j=0}^{n-1} z^j = 0$$

$$\langle g_s, g_t \rangle = 0$$

provided that $s \in \{0, \ldots, n-1\}$

$t \in \{0, \ldots, n-1\}$

$s \neq t$

$$\|g_s\|^2 = \langle g_s, g_s \rangle$$

$$= \sum_x e^{-2\pi i \frac{x \cdot s}{n}} \cdot e^{+2\pi i \frac{x \cdot s}{n}}$$

$$= \sum_x 1 = n$$

$$f(x) = \sum_s \hat{f}(s)\, \hat{g}_s(x)$$

$$\hat{f}(s) = \frac{1}{n} \underbrace{\langle f, g_s \rangle}$$

given $f(0), f(1), f(2), \ldots, f(n-1)$

want to compute

$$\hat{f}(0) = \sum_j f(j) \cdot 1 = \sum_j f(j)$$

$$\hat{f}(1) = \sum_j f(j) \cdot e^{\frac{2\pi i}{n} \cdot j} = \sum_j f(j) \cdot w^j$$

$$\vdots$$

$$\hat{f}(s) = \sum_j f(j) \cdot e^{\frac{2\pi i}{n} \cdot s \cdot j} = \sum_j f(j) \cdot w^{s \cdot j}$$

$$\vdots$$

$$\hat{f}(n-1) = \sum_j f(j) \cdot e^{\frac{2\pi i}{n} \cdot (n-1) \cdot j} = \sum_j f(j) w^{(n-1) j}$$

$$w_n = e^{2\pi i/n}$$

$$\begin{array}{c c}
 & k \\
j & \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1, & w, & w^2 \cdots & w^{n-1} \\ 1 & w^2 & w^4 \cdots, & w^{2\cdot(n-1)} \\ & & w^{k\cdot j} & \\ & & & \end{pmatrix}
\end{array}
\begin{pmatrix} f(0) \\ f(1) \\ \vdots \\ f(n-1) \end{pmatrix} = \begin{pmatrix} f(0)+\cdots+f(n-1) \\ f(0)+wf(1) \\ \\ f(0)+w^{n-1}f(1)+\cdots \end{pmatrix}$$

$$k = 0, \ldots, n-1$$

$$j = 0, \ldots, n-1$$

Define polynomial $p(x)$

$$p(x) = f(0) + f(1)x + f(2)x^2 + \cdots + f(n-1)x^{n-1}$$

$$p(1) = \hat{f}(0)$$

$$p(w) = \hat{f}(1)$$

$$p(w^2) = \hat{f}(2)$$

$$\vdots$$

$$p(w^{n-1}) = \hat{f}(n-1)$$

$$\boxed{\begin{aligned} &\text{given} \quad p(x) = \sum_{j=0}^{n-1} f(j)\, x^j \\ &\omega = e^{2\pi i/n} \\ &\text{compute} \quad p(\omega_n^0),\, p(\omega_n^1),\, --,\, p(\omega^{n-1}) \end{aligned}}$$

Assume

$n$ power of 2

$$p(x) = \overbrace{f(0) + f(2)x^2 + f(4)x^4 + \cdots + f(n-2)x^{n-2}}$$
$$+ \underbrace{f(1)x + f(3)x^3 + - - - - + f(n-2)x^{n-1}}$$

define $P_{even}(y) = f(0) + f(2)y + f(4)y^2 + \cdots f(n-2)y^{\frac{n}{2}-1}$

then $P_{even}(x^2) = f(0) + f(2)x^2 + - - - + f(n-2)x^2$

evaluate $\boxed{P_{even}(y) \quad y = \omega^0_{n/2},\, \omega^1_{\frac{n}{2}},\, \omega^2_{n/2} \,--,\, \omega^{n/2-1}_{n/2}}$

$$\omega_{n/2} = e^{2\pi i/n/2} = e^{\frac{2\pi i}{n}\cdot 2} = \omega_n^2$$

$$\omega^j_{n/2} = e^{\frac{2\pi i}{n}\cdot 2j} = \omega_n^{2j}$$

$$P_{odd}(y) = f(1) + f(3)y + f(5)y^2 + \cdots f(n-1)y^{\frac{n}{2}-1}$$

$$f(1)x + f(3)x^3 + \cdots + f(n-1)x^{n-1}$$
$$= x \cdot \boxed{P_{odd}(x^2)} \qquad \text{for} \quad x = \omega_n^0, \omega_n^1, --, \omega_n^{n-1}$$

Define $FFT(n, f(0), \ldots, f(n-1))$

output: $f(0) + \cdots + f(n-1)$,

$f(0) + w_n f(1) + \cdots w_n^{n-1} f(n-1)$,

$f(0) + w_n^2 f(1) + \cdots w_n^{2(n-1)} f(n-1)$. $\quad w_n = e^{\frac{2\pi i}{n}}$

$\vdots$

$f(0) + w_n^{n-1} f(1) + \cdots w_n^{(n-1)\cdot(n-1)} f(n-1)$.

$FFT(n, f(0) \ldots f(n-1))$

if $n == 1$ return $f(0)$

else

$(a_0, \ldots, a_{\frac{n}{2}-1}) = FFT(\frac{n}{2}, f(0), f(2) \ldots f(n-2))$

$(b_0, \ldots b_{\frac{n}{2}-1}) = FFT(\frac{n}{2}, f(1), f(3) \ldots f(n-1))$

return $\Big( a_0 + w_n^0 b_0, a_1 + w_n^1 b_1, \ldots, a_{\frac{n}{2}-1} + w_n^{\frac{n}{2}-1} b_{\frac{n}{2}-1},$

$a_0 + w_n^{n/2} b_0, a_1 + w_n^{\frac{n}{2}+1} b_1, \ldots, a_{\frac{n}{2}-1} + w_n^{n-1} b_{\frac{n}{2}-1} \Big)$

$n=4$

$f(0) = 2$

$f(1) = 3$

$f(2) = 4$

$f(3) = 1$


$$w_4 = e^{2\pi i/4} = e^{\pi i/2} = i$$

$$\hat{f}(0) = \overset{2}{f(0)} + \overset{3}{f(1)} + \overset{4}{f(2)} + \overset{1}{f(3)} = 10$$

$$\hat{f}(1) = \overset{2}{f(0)} + \overset{+3i}{i \cdot f(1)} + \overset{-4}{i^2 f(2)} + \overset{-i}{i^3 f(3)}$$

$$\hat{f}(2) = \overset{2}{f(0)} - \overset{-3}{f(1)} + \overset{+4}{f(2)} - \overset{-1}{f(3)}$$

$$\hat{f}(3) = \overset{2}{f(0)} - \overset{-3i}{i f(1)} + \overset{-4}{i^6 f(2)} + \overset{+i}{i^9 f(3)}$$

$6+4, -2+2i, 6-4, -2+i^3 2$ $\qquad$ $10, -2+2i, 2, -2-2i$

$f(0)$ $f(1)$ $f(2)$ $f(3)$ $\qquad w_n = e^{2\pi i/n}$ $\qquad n$

$2, 3, 4, 1$ $\qquad i$ $\qquad 4$

$6, -2$ $\qquad 4, 2$

$2+4, 2-4$ $\qquad w_n = e^{2\pi i/n}$

$2, 4$ $\quad -1$ $\qquad 3, 1$ $\quad -1$

$2 \quad 4$ $\qquad 3 \quad 1$

$2 \quad 4 \quad 1$ $\qquad 3 \quad 1 \quad 1$

$$q(x) = 2 + 4x$$

$$r(x) = 3 + x$$

$$p(x) = 2 + 3x + 4x^2 + x^3$$

$$p(x) = \underbrace{\underbrace{q(x^2)} + x \cdot r(x^2)}$$

$$q(y) \qquad y = 1, -1$$

$$r(y) \qquad y = 1, -1$$

$$p(1) = q(1^2) + 1 \cdot r(1^2)$$

$$p(i) = q(-1) + i \cdot r(-1)$$

$$p(i^2) = q(1) + i^2 r(1)$$

$$p(i^3) = q(-1) + i^3 r(-1)$$

$x = 1, i, -1, -i$

$x = 1, i, -1, -i$

$x^2 = 1, -1, 1, -1$

$T(n)$ time FFT algorithm

$n$ dimension

$$T(n) = 2T(n/2) + O(n)$$

$$T(1) = O(1)$$

$$T(n) = O(n \log n)$$

Main points:

- idea of divide-and-conquer
- master theorem

$$T(n) = a\,T(n/b) + n^d$$

- $$T(n) = 2\,T(n-1) + O(n)$$

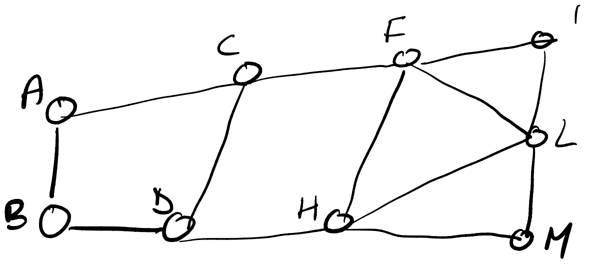
exponential time

$$T(n) = T(n-1) + O(n)$$

$$O(n^2)$$

# Graphs / Networks

Representation of relations between <u>pairs</u>




Specified by

a finite set V of vertices

$$V = \{A, B, C, D, F, H, I, L, M\}$$

a finite set E of pairs of vertices

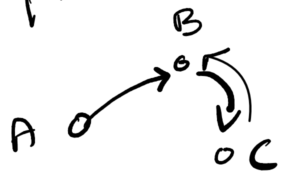
$$E = \{(A,B), (A,C), (B,D), \ldots\}$$

Undirected Graphs:

E is a set of unordered pairs

Directed graph

E is a set of ordered pairs

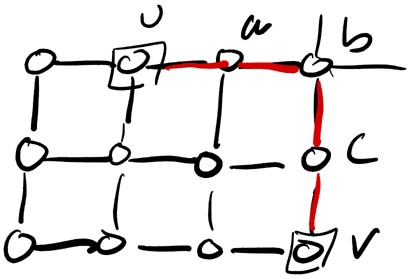



$V = \{A, B, C \ldots$

$E = \{(A,B), (B,C), (C,B), \ldots$

Vertices: street intersections

Edge: piece of road between two intersections




A path in a graph from vertex $u$ to vertex $v$ is a sequence of nodes

$u, a, b, c, v$

$(u,a)$ $(a,b)$ $(b,c)$ $(c,v)$

that starts at $u$, ends at $v$ and there is an edge between any two consecutive nodes in the sequence

Given a graph are there paths between any two nodes?