$\bar{u}_i \in V'$. To see that this truth assignment satisfies each of the clauses $c_j \in C$, consider the three edges in E_j''. Only two of those edges can be covered by vertices from $V_j' \cap V'$, so one of them must be covered by a vertex from some V_i that belongs to V'. But that implies that the corresponding literal, either u_i or $\bar{u}_i$, from clause c_j is true under the truth assignment t, and hence clause c_j is satisfied by t. Because this holds for every $c_j \in C$, it follows that t is a satisfying truth assignment for C.

Conversely, suppose that $t: U \to \{T,F\}$ is a satisfying truth assignment for C. The corresponding vertex cover V' includes one vertex from each T_i and two vertices from each S_j. The vertex from T_i in V' is u_i if $t(u_i) = T$ and is $\bar{u}_i$ if $t(u_i) = F$. This ensures that at least one of the three edges from each set E_j'' is covered, because t satisfies each clause c_j. Therefore we need only include in V' the endpoints from S_j of the other two edges in E_j'' (which may or may not also be covered by vertices from truth-setting components), and this gives the desired vertex cover. ■

3.1.4 HAMILTONIAN CIRCUIT

In Chapter 2, we saw that the HAMILTONIAN CIRCUIT problem can be transformed to the TRAVELING SALESMAN decision problem, so the NP-completeness of the latter problem will follow immediately once HC has been proved NP-complete. At the end of the proof we note several variants of HC whose NP-completeness also follows more or less directly from that of HC.

For convenience in what follows, whenever $<v_1, v_2, \ldots, v_n>$ is a Hamiltonian circuit, we shall refer to $\{v_i, v_{i+1}\}$, $1 \leqslant i < n$, and $\{v_n, v_1\}$ as the edges "in" that circuit. Our transformation is a combination of two transformations from [Karp, 1972], also described in [Liu and Geldmacher, 1978].

Theorem 3.4 HAMILTONIAN CIRCUIT is NP-complete

Proof: It is easy to see that HC $\in$ NP, because a nondeterministic algorithm need only guess an ordering of the vertices and check in polynomial time that all the required edges belong to the edge set of the given graph.

We transform VERTEX COVER to HC. Let an arbitrary instance of VC be given by the graph $G = (V,E)$ and the positive integer $K \leqslant |V|$. We must construct a graph $G' = (V',E')$ such that G' has a Hamiltonian circuit if and only if G has a vertex cover of size K or less.

Once more our construction can be viewed in terms of components connected together by communication links. First, the graph G' has K "selector" vertices $a_1, a_2, \ldots, a_K$, which will be used to select K vertices from the vertex set V for G. Second, for each edge in E, G' contains a "cover-testing" component that will be used to ensure that at least one endpoint of that edge is among the selected K vertices. The component for

$e=\{u,v\} \in E$ is illustrated in Figure 3.4. It has 12 vertices,

$$V'_e = \{(u,e,i),(v,e,i): 1 \leqslant i \leqslant 6\}$$

and 14 edges,

$$\begin{aligned} E'_e = & \{\{(u,e,i),(u,e,i+1)\},\{(v,e,i),(v,e,i+1)\}: 1 \leqslant i \leqslant 5\} \\ & \cup \{\{(u,e,3),(v,e,1)\},\{(v,e,3),(u,e,1)\}\} \\ & \cup \{\{(u,e,6),(v,e,4)\},\{(v,e,6),(u,e,4)\}\} \end{aligned}$$

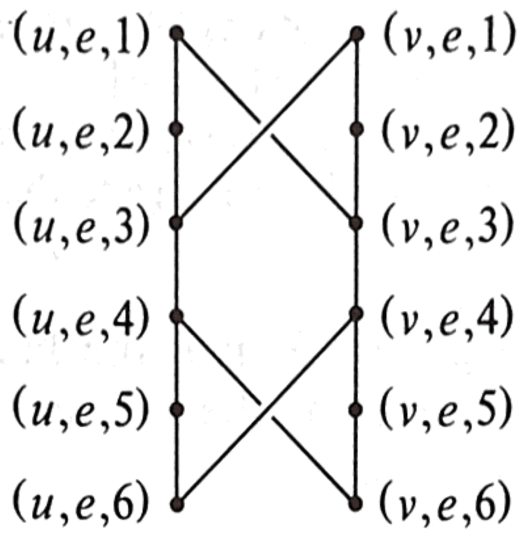

Figure 3.4 Cover-testing component for edge $e=\{u,v\}$ used in transforming VERTEX COVER to HAMILTONIAN CIRCUIT.

In the completed construction, the only vertices from this cover-testing component that will be involved in any additional edges are $(u,e,1)$, $(v,e,1)$, $(u,e,6)$, and $(v,e,6)$. This will imply, as the reader may readily verify, that any Hamiltonian circuit of G' will have to meet the edges in E'_e in exactly one of the three configurations shown in Figure 3.5. Thus, for example, if the circuit "enters" this component at $(u,e,1)$, it will have to "exit" at $(u,e,6)$ and visit either all 12 vertices in the component or just the 6 vertices (u, e, i), $1 \leqslant i \leqslant 6$.

Additional edges in our overall construction will serve to join pairs of cover-testing components or to join a cover-testing component to a selector vertex. For each vertex $v \in V$, let the edges incident on v be ordered (arbitrarily) as $e_{v[1]}, e_{v[2]}, \ldots, e_{v[deg(v)]}$, where $deg(v)$ denotes the *degree* of v in G, that is, the number of edges incident on v. All the cover-testing components corresponding to these edges (having v as endpoint) are joined together by the following connecting edges:

$$E'_v = \{\{(v,e_{v[i]},6),(v,e_{v[i+1]},1)\}: 1 \leqslant i < deg(v)\}$$

As shown in Figure 3.6, this creates a single path in G' that includes exactly those vertices (x,y,z) having $x=v$.

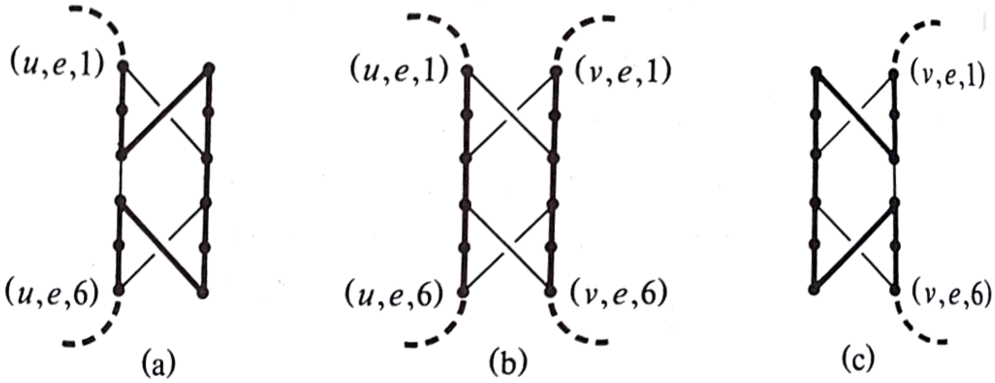

Figure 3.5 The three possible configurations of a Hamiltonian circuit within the cover-testing component for edge $e=\{u,v\}$, corresponding to the cases in which (a) u belongs to the cover but v does not, (b) both u and v belong to the cover, and (c) v belongs to the cover but u does not.

The final connecting edges in G' join the first and last vertices from each of these paths to every one of the selector vertices $a_1, a_2, \ldots, a_K$. These edges are specified as follows:

$$E'' = \{\{a_i,(v,e_{v[1]},1)\},\{a_i,(v,e_{v[deg(v)]},6)\}: 1 \leqslant i \leqslant K,\ v \in V\}$$

The completed graph $G' = (V',E')$ has

$$V' = \{a_i: 1 \leqslant i \leqslant K\} \cup (\bigcup_{e \in E} V'_e)$$

and

$$E' = (\bigcup_{e \in E} E'_e) \cup (\bigcup_{v \in V} E'_v) \cup E''$$

It is not hard to see that G' can be constructed from G and K in polynomial time.

We claim that G' has a Hamiltonian circuit if and only if G has a vertex cover of size K or less. Suppose $<v_1, v_2, \ldots, v_n>$, where $n = |V'|$, is a Hamiltonian circuit for G'. Consider any portion of this circuit that begins at a vertex in the set $\{a_1, a_2, \ldots, a_K\}$, ends at a vertex in $\{a_1, a_2, \ldots, a_K\}$, and that encounters no such vertex internally. Because of the previously mentioned restrictions on the way in which a Hamiltonian circuit can pass through a cover-testing component, this portion of the circuit must pass through a set of cover-testing components corresponding to exactly those edges from E that are incident on some one particular vertex $v \in V$. Each of the cover-testing components is traversed in one of the modes (a), (b), or (c) of Figure 3.5, and no vertex from any other cover-testing component is encountered. Thus the K vertices from $\{a_1, a_2, \ldots, a_K\}$ divide the Hamiltonian circuit into K paths, each path

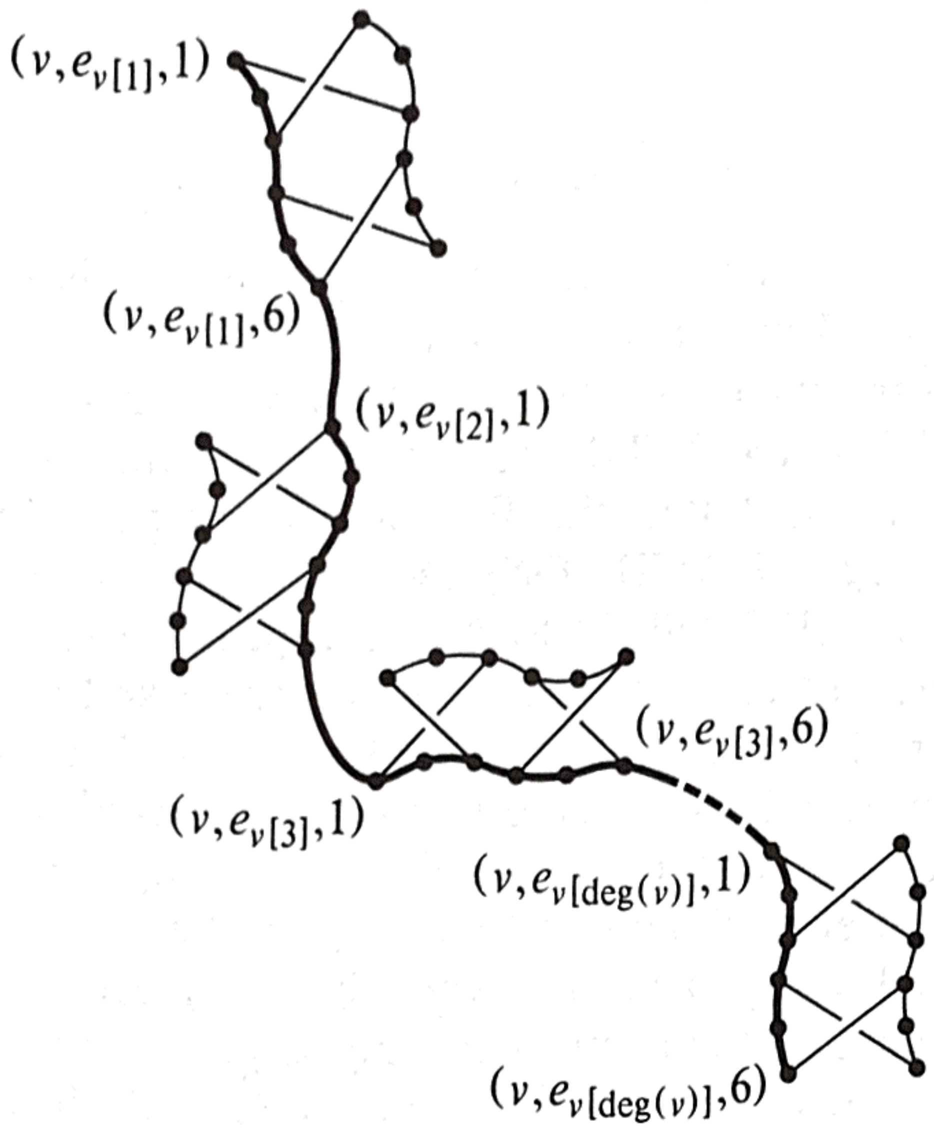

Figure 3.6 Path joining all the cover-testing components for edges from E having vertex v as an endpoint.

corresponding to a distinct vertex $v \in V$. Since the Hamiltonian circuit must include all vertices from every one of the cover-testing components, and since vertices from the cover-testing component for edge $e \in E$ can be traversed only by a path corresponding to an endpoint of e, every edge in E must have at least one endpoint among those K selected vertices. Therefore, this set of K vertices forms the desired vertex cover for G.

Conversely, suppose $V^* \subseteq V$ is a vertex cover for G with $|V^*| \leqslant K$. We can assume that $|V^*| = K$ since additional vertices from V can always be added and we will still have a vertex cover. Let the elements of V^* be labeled as $v_1, v_2, \ldots, v_K$. The following edges are chosen to be "in" the Hamiltonian circuit for G'. From the cover-testing component representing each edge $e = \{u, v\} \in E$, choose the edges specified in Figure 3.5(a), (b), or (c) depending on whether $\{u, v\} \cap V^*$ equals, respectively, $\{u\}$, $\{u, v\}$, or $\{v\}$. One of these three possibilities must hold since V^* is a vertex cover for G. Next, choose all the edges in E'_{v_i} for $1 \leqslant i \leqslant K$. Finally, choose the edges

$$\{a_i, (v_i, e_{v_i[1]}, 1)\}, 1 \leqslant i \leqslant K$$

$$\{a_{i+1}, (v_i, e_{v_i[deg(v_i)]}, 6)\}, 1 \leq i < K$$

and

$$\{a_1, (v_K, e_{v_K[deg(v_K)]}, 6)\}$$

We leave to the reader the task of verifying that this set of edges actually corresponds to a Hamiltonian circuit for G'. ■

Several variants of HAMILTONIAN CIRCUIT are also of interest. The HAMILTONIAN PATH problem is the same as HC except that we drop the requirement that the first and last vertices in the sequence be joined by an edge. HAMILTONIAN PATH BETWEEN TWO POINTS is the same as HAMILTONIAN PATH, except that two vertices u and v are specified as part of each instance, and we are asked whether G contains a Hamiltonian path beginning with u and ending with v. Both of these problems can be proved NP-complete using the following simple modification of the transformation just used for HC. We simply modify the graph G' obtained at the end of the construction as follows: add three new vertices, a_0, a_{K+1}, and a_{K+2}, add the two edges $\{a_0, a_1\}$ and $\{a_{K+1}, a_{K+2}\}$, and replace each edge of the form $\{a_1, (v, e_{v[deg(v)]}, 6)\}$ by $\{a_{K+1}, (v, e_{v[deg(v)]}, 6)\}$. The two specified vertices for the latter variation of HC are a_0 and a_{K+2}.

All three Hamiltonian problems mentioned so far also remain NP-complete if we replace the undirected graph G by a directed graph and replace the undirected Hamiltonian circuit or path by a directed Hamiltonian circuit or path. Recall that a directed graph $G=(V,A)$ consists of a vertex set V and a set of *ordered* pairs of vertices called *arcs*. A Hamiltonian path in a directed graph $G=(V,A)$ is an ordering of V as $<v_1, v_2, \ldots, v_n>$, where $n=|V|$, such that $(v_i, v_{i+1}) \in A$ for $1 \leq i < n$. A Hamiltonian circuit has the additional requirement that $(v_n, v_1) \in A$. Each of the three undirected Hamiltonian problems can be transformed to its directed counterpart simply by replacing each edge $\{u,v\}$ in the given undirected graph by the two arcs (u,v) and (v,u). In essence, the undirected versions are merely special cases of their directed counterparts.

3.1.5 PARTITION

In this section we consider the last of our six basic NP-complete problems, the PARTITION problem. It is particularly useful for proving NP-completeness results for problems involving numerical parameters, such as lengths, weights, costs, capacities, etc.

Theorem 3.5 PARTITION is NP-complete
Proof: It is easy to see that PARTITION $\in$ NP, since a nondeterministic algorithm need only guess a subset A' of A and check in polynomial time