Prove the Max Flow Min Cut Thm

proof of correctness of Ford Fulkerson regardless of how we choose path in each iteration

Running Time Analysis of Edmonds-Karp

Ford-Fulkerson alg with BFS

Applications of Max Flow

Theorem

Max Flow Mincut Theorem

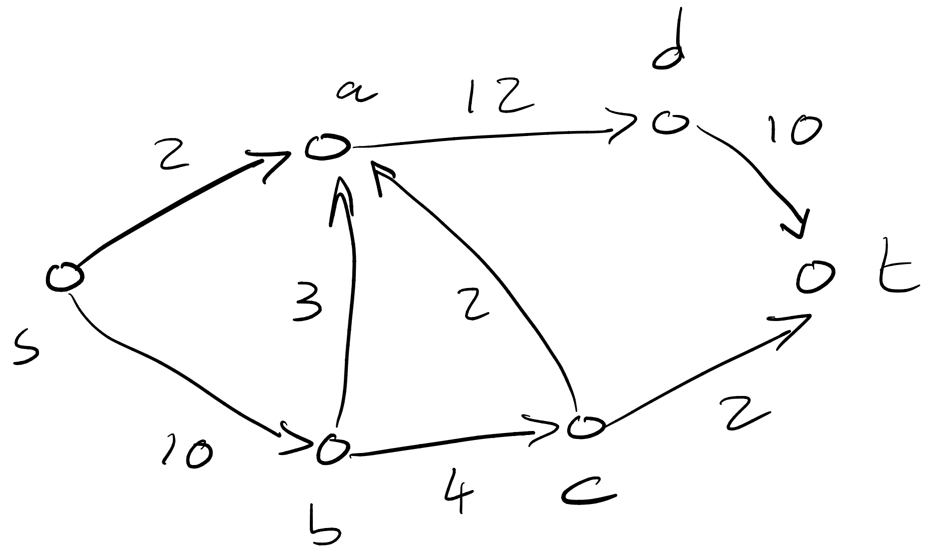
Let N be a network
Let f be the output of Ford-Fulkerson
Let S be the set of vertices reachable from s in the residual network of N with respect to flow f

Then $\text{cap}(S) = \text{val}(f)$

For any other flow f' $\text{val}(f') \leq \text{cap}(S) = \text{val}(f)$
and so f is optimal

For every cut S, every flow f

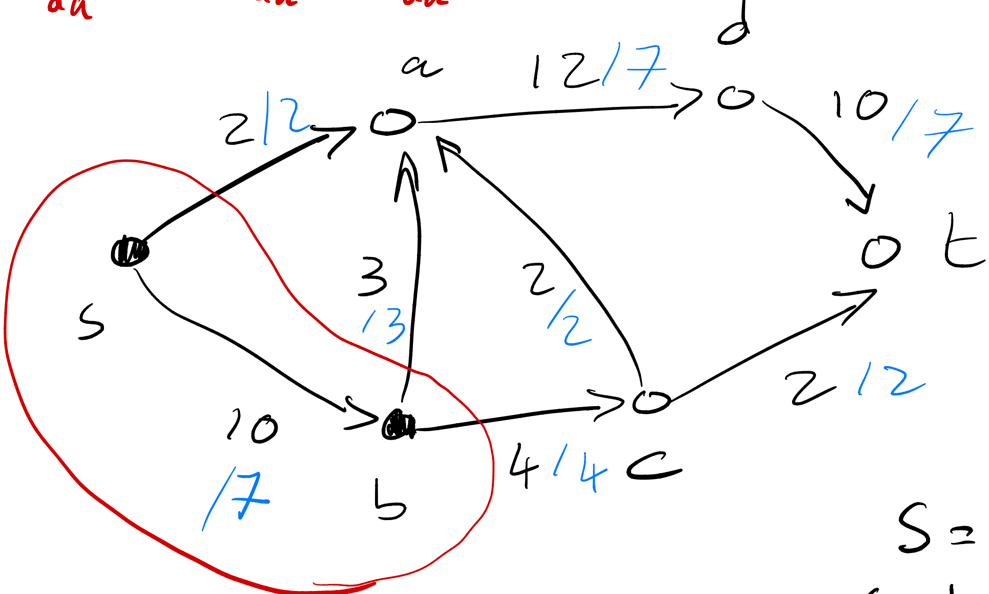
$$\mathrm{val}(f) = \sum_{u \in S} \sum_{v \notin S} f_{uv} \leq \sum_{u \in S} \sum_{v \notin S} c_{u,v}$$



$r_{ad} = c_{ad} - f_{ad} = 12 - 7 = 5$

$r_{da} = c_{da} - f_{da} = 0 - (-7) = 7$

$S = \{s, c\}$



c_{uv}

$f_{u,v}$

$\boxed{r_{u,v} = c_{uv} - f_{u,v}}$

$S = \{s, b\}$

$S = \{v : v$ reachable from s in residual network$\}$

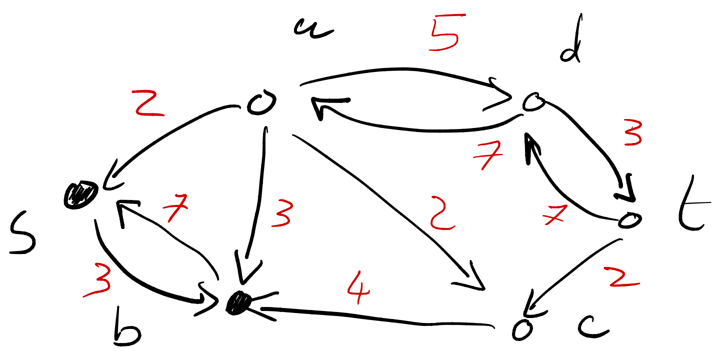
$\mathrm{cap}(S) = 2 + 3 + 4 = 9$

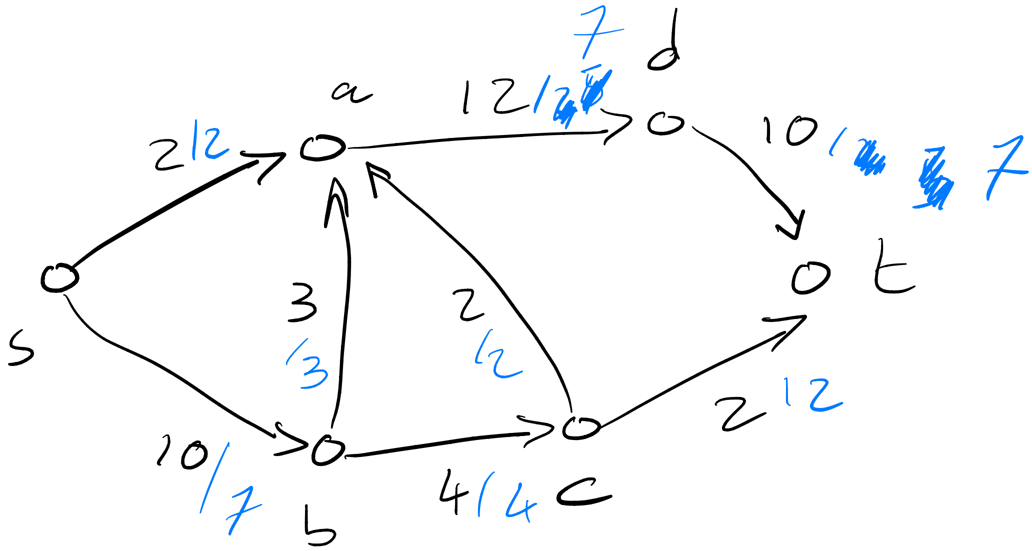
$= \mathrm{val}(f)$

$c_{as} = 0$

$f_{as} = -2$

$= -f_{sa}$

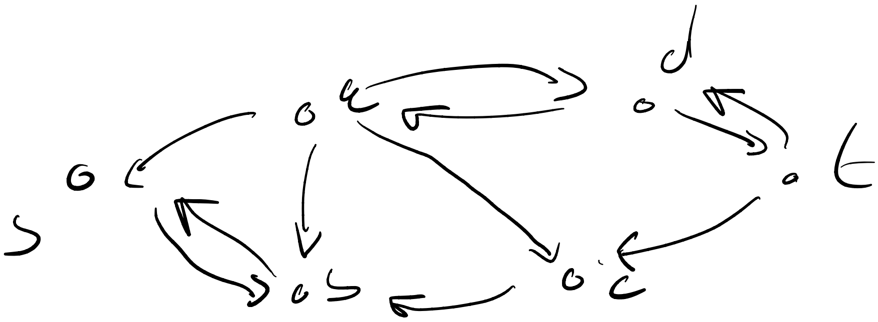




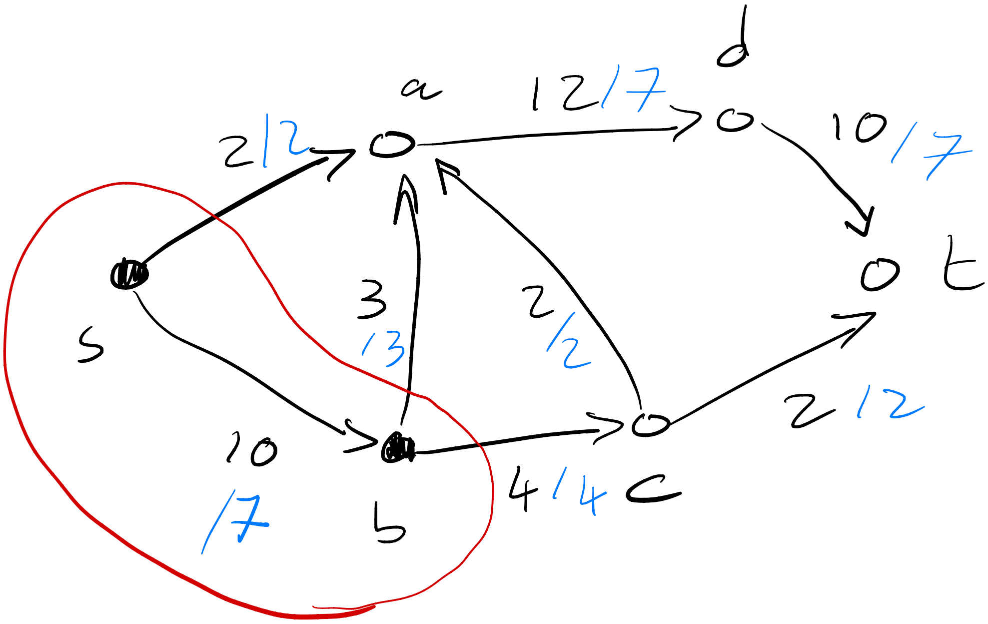
distances from s in residual network

s	a	b	c	d	t
0	∞	1	∞	∞	∞

$S = \{s, b\}$



$S = \{s, c\}$



$\mathrm{cap}(\{s, c\}) = 2 + 10 + 2 + 2$
$= 16$

$S = \{s, c\}$ $\qquad \sum_{u \in S} \sum_{v \notin S} f_{u,v} = 9$

In general

f output of algorithm

$S = \{v : v$ reachable from s in residual network with respect to $f\}$

$s \in S \quad t \notin S$

S is a cut

consider any (a,b) $\quad a \in S \quad b \notin S$

claim: $f_{a,b} = c_{a,b}$

proof: if $f_{a,b} < c_{a,b}$ then (a,b) belongs to residual network

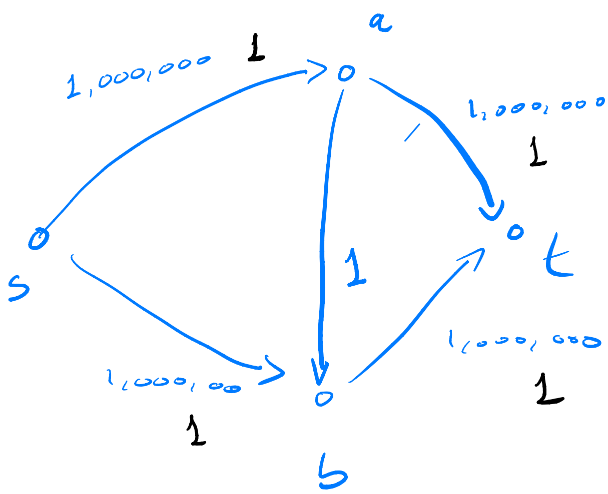
a path $s \rightsquigarrow a$ exists in residual network because $a \in S$

but then $s \rightsquigarrow a \to b$ is a path from s to b in residual network

but $b \notin S$ so such a path cannot exist

$$val(f) = \sum_{a \in S} \sum_{b \notin S} f_{a,b} = \sum_{a \in S} \sum_{b \notin S} c_{a,b}$$

$$= cap(S)$$


a
s
t
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1,000,000
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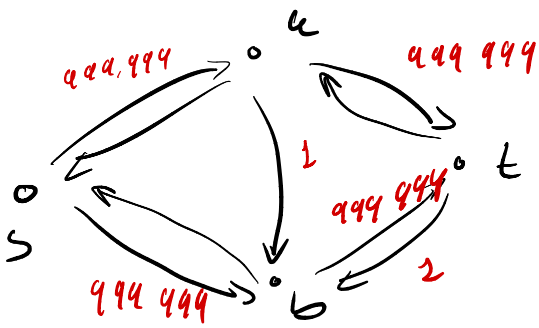
1
1,000,000
1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999 999
999 999
1

Edmonds-Karp algorithm

Ford-Folkerson, with BFS used to find path in residual network

Each iteration

$O(|E|)$ to find path

$O(|V|)$ to update flow
update residual network

How many iterations?

Main Thm of Edmonds-Karp
There can be at most $|V| \cdot |E|$ iterations

structure of proof

1. Distance of nodes from s in residual network can only increase and can be at most $|V| - 1$

2. If an edge (u,v) is bottleneck of augmenting path at some iteration, and distance of u from s is ℓ at that iteration, then next time (u,v) is bottleneck of augmenting path we have that distance from s to u is $\geq \ell + 2$

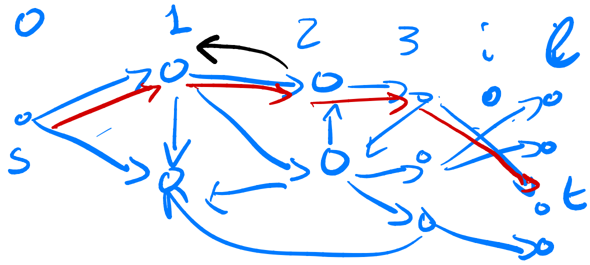
   (u,v) can be bottleneck edge $\leq \frac{|V|-1}{2}$ times

3. There are $\leq 2|E|$ pairs (u,v) that can be a bottleneck edge

\# iterations $\leq 2 \cdot |E| \cdot (|V|-1)/2 < |E| \cdot |V|$

distances from s in residual network
after k iterations

nodes at distance



path p

residual network after $k+1$ iterations
same as residual network at time k except

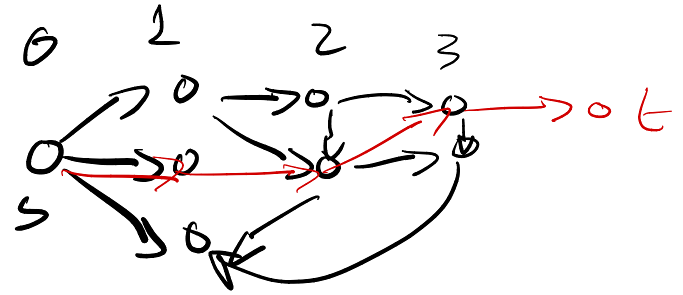
- some edges (u,v) of p might have disappeared
- some edges (w,z) such that (z,w) in p might be added

consider a vertex v at distance i from s at iteration k

what about paths from s to v after iteration $k+1$

call $d_k(s,a)$ distance from s to a after k iteration

if an edge (u,v) exists in residual network after $k+1$ iterations $d_k(s,v) \leq 1+d_k(s,u)$



For every edge (a,b) in network at iteration k

$$d_k(s,b) \leq 1 + d_k(s,a)$$

$$d_k(s,v) = i$$

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network at iteration $k+1$

every edge (a,b) is either an edge that was in network at step k

or an edge $d_k(s,b) = d_k(s,a) - 1$

either way

$$d_k(s,b) \leq 1 + d_k(s,a)$$

take any path from s to v in network of iteration $k+1$

$$s \to v_1 \to v_2 \to \ldots \to v_{\ell-1} \to v$$

$$d_k(s, v_1) \leq 1 + d_k(s,s) = 1$$

$$d_k(s, v_2) \leq 1 + d_k(s, v_1) \leq 2$$

$$\vdots$$

$$i = d_k(s,v) \leq \ell$$

number of steps to go from s to v after iteration $k+1$ $\geq$ number of steps after it. k

(2) Suppose (u,v) is a bottleneck at iteration k, and, in that iteration $d_k(s,u) = \ell$

Suppose (u,v) is again a bottleneck at a later iteration $k' > k$

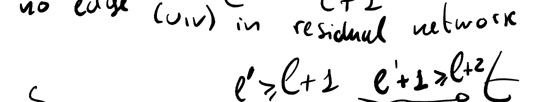
Then $d_{k'}(s,u) \geq \ell + 2$

Proof

- at iteration k $d_k(s,v) = \ell + 1$
- after iteration k, (u,v) disappears from residual network
- at some iteration $k'' < k'$, (u,v) reappear in residual network.
  - $\rightarrow$ (v,u) was in the shortest path from s to t in iteration k''

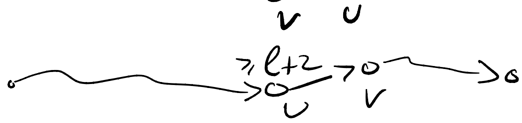
$$d_{k'}(s,u) \geq d_{k''}(s,u) = 1 + d_{k''}(s,v)$$
$$\geq 1 + \ell + 1$$
$$= \ell + 2$$

time k

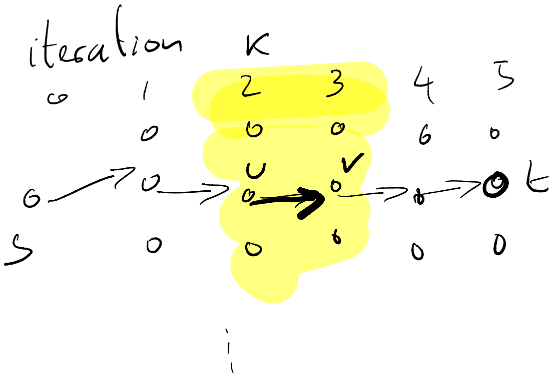


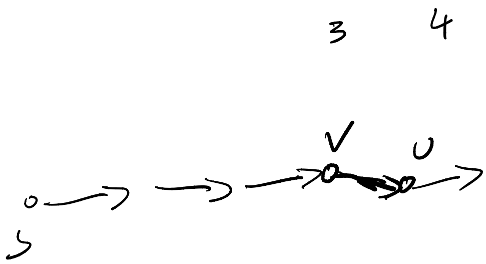
time $k+1$ there is no edge (u,v) in residual network

time k''



time k'


iteration k
0 1 2 3 4 5
u
v
t
s


3 4
v
u
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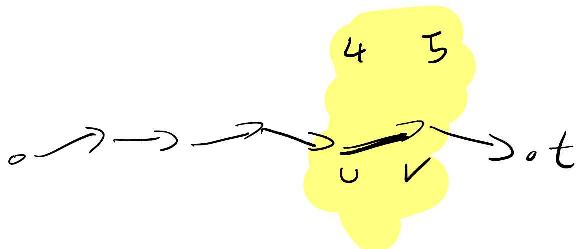
4 5
u
v
t

Each pair (u,v) can be a bottleneck $\leq \frac{|V|-1}{2}$ times

because distance of u from s is ≥ 0 at beginning

$\leq |V|-1$ at the end

Number of pairs (u,v) that can be a bottleneck is $\leq 2 \cdot |E|$

Every iteration uses at least one (u,v) as a bottleneck

Number of iterations $\leq 2|E| \cdot \left(\frac{|V|-1}{2}\right) < |E| \cdot |V|$