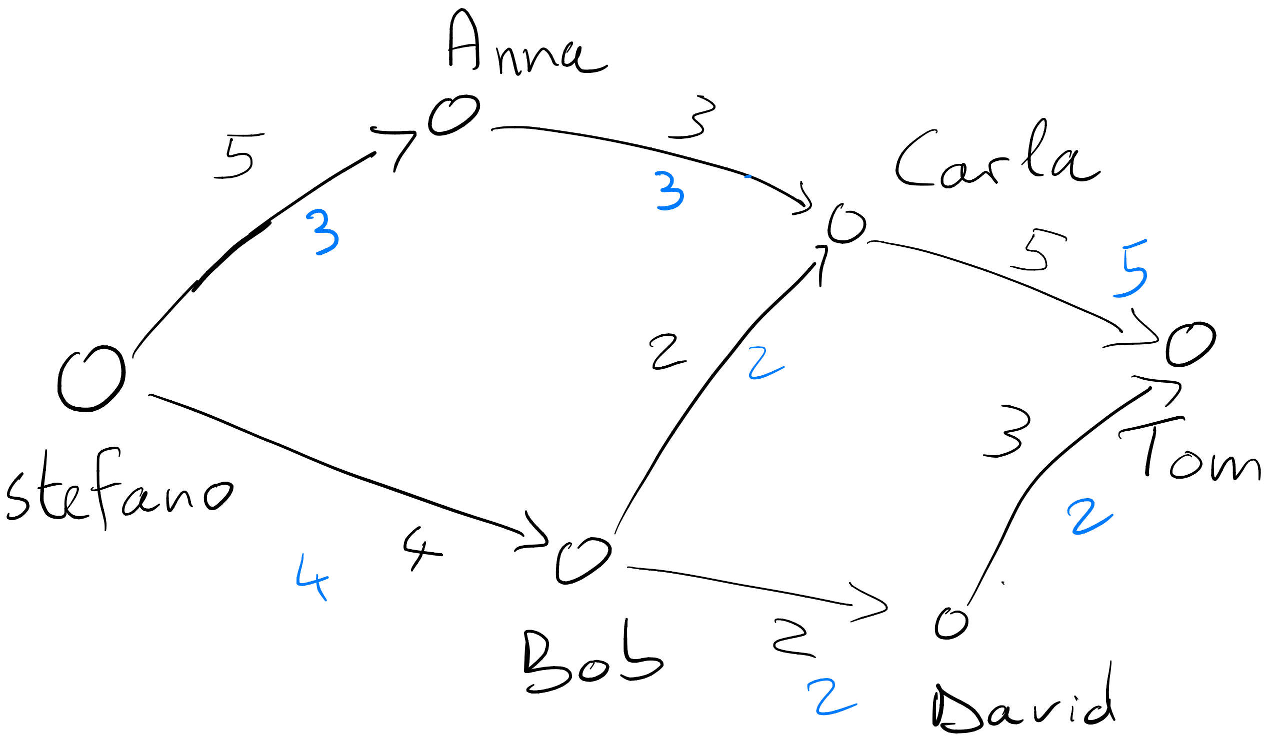
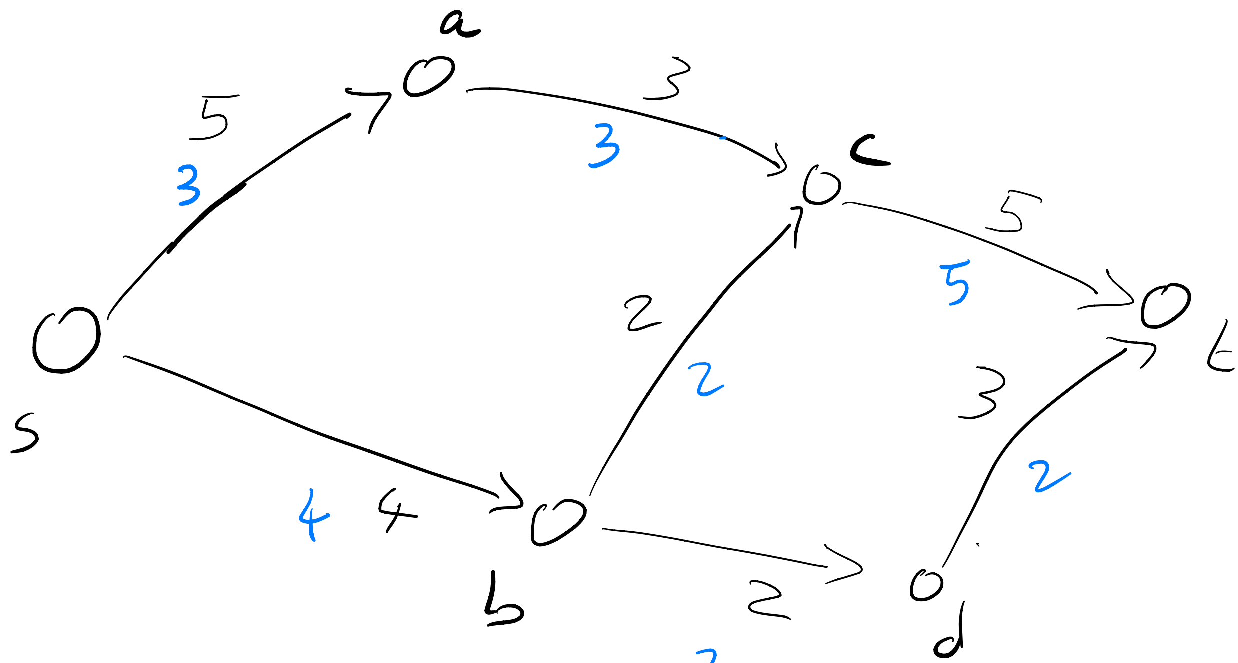
Flow In Networks

Peer-to-peer Lending



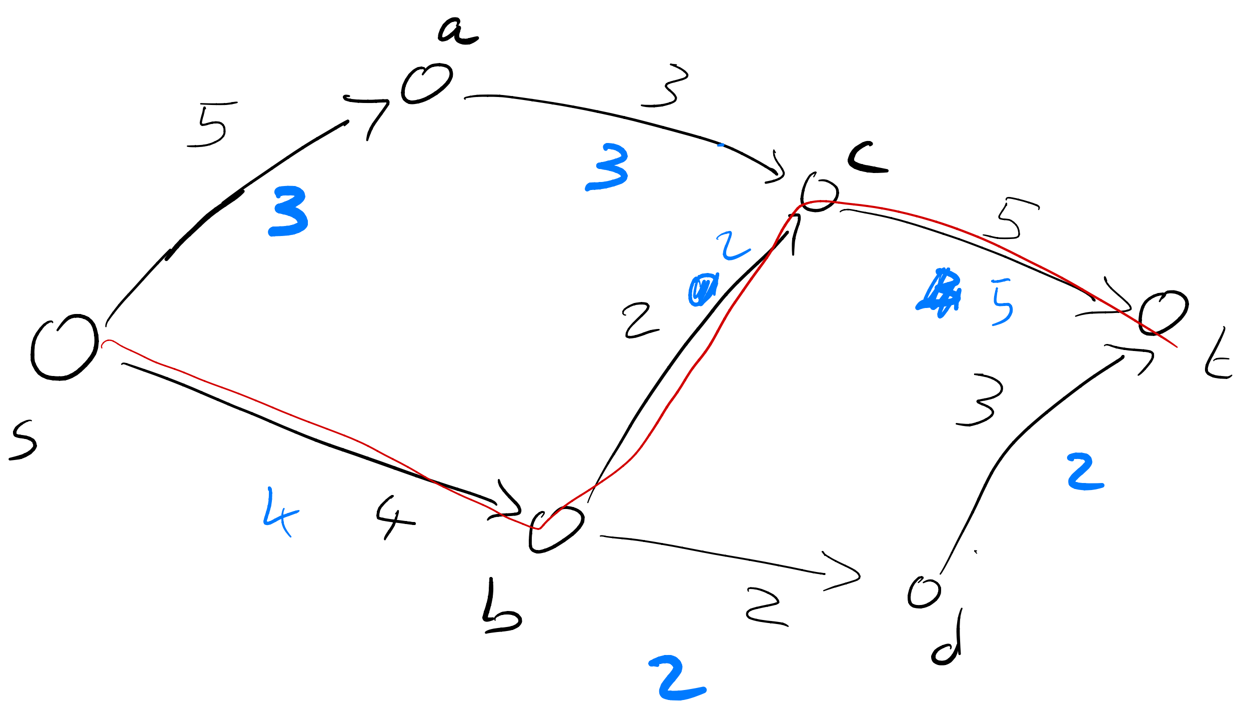
Trust in K €

Throughput in Network



Bandwidth in Gb/s

Max Flow Problem Def.



Given:

network → $\begin{cases} \cdot \text{ directed graph } G=(V,E) \\ \cdot \text{ capacity } c_{u,v} \geq 0 \text{ for every edge } (u,v) \in E \\ \cdot \text{ a sender } s \in V \text{ and a receiver } t \in V \end{cases}$

Want to compute
<u>flow</u> of maximum <u>value</u>

A <u>flow</u> in a network is an assignment of a value $f_{u,v}$ to each edge (u,v)

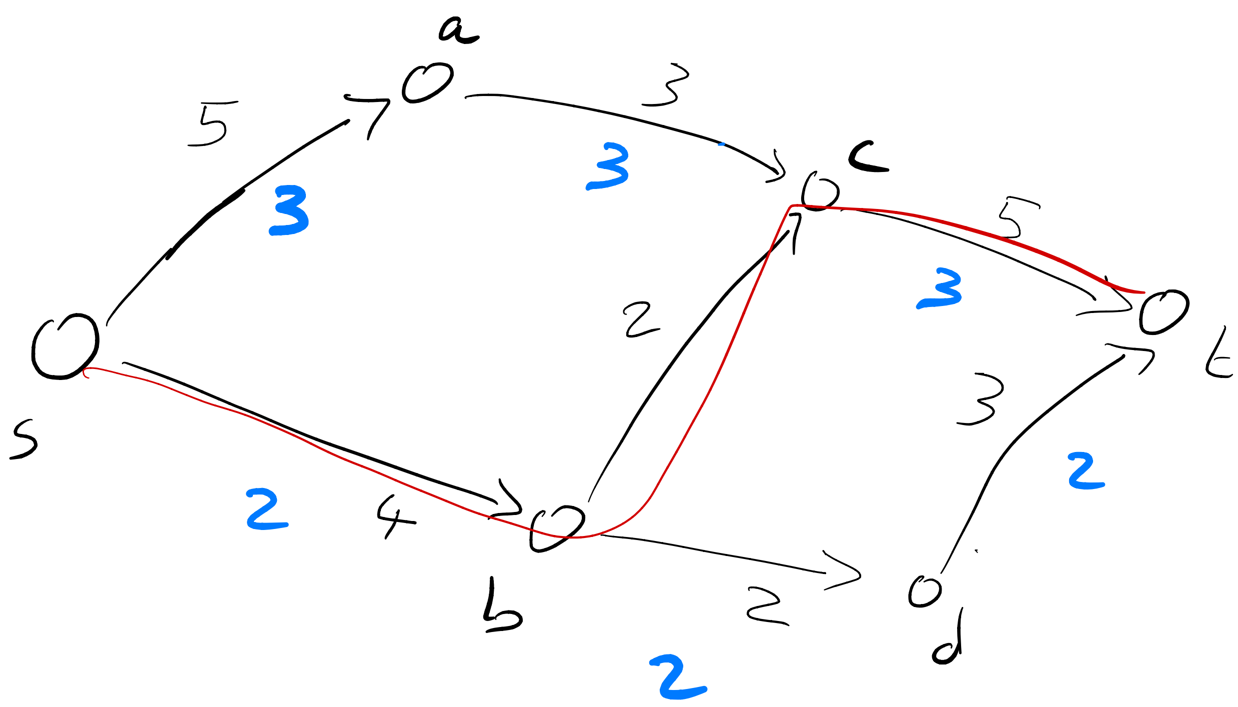
- $0 \leq f_{u,v} \leq c_{u,v}$ capacity
- $\forall v \in V \quad v \neq s, v \neq t$ conservation

$$\sum_{u:(u,v)\in E} f_{u,v} = \sum_{w:(v,w)\in E} f_{v,w}$$

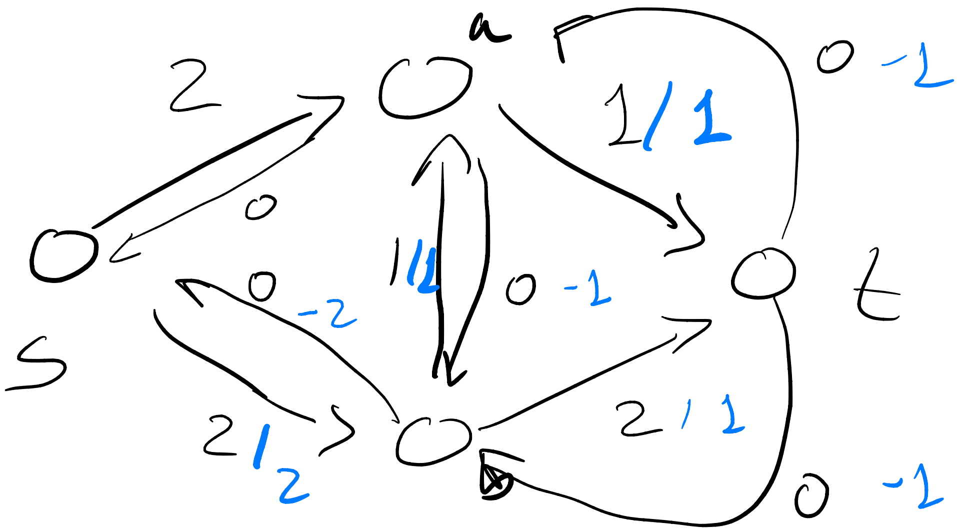
value of a flow f is

$$\mathrm{val}(f) = \sum_{v:(s,v)} f_{s,v}$$

Augmenting Path



Augmenting Path

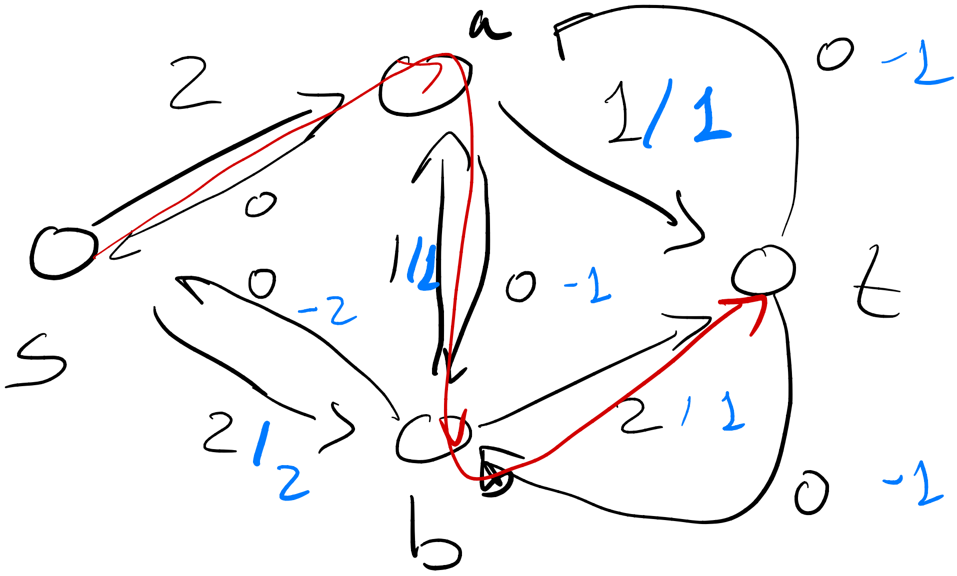


Def: flow is an assignment $f_{u,v}$ to every u,v such that (u,v) or (v,u) is in E

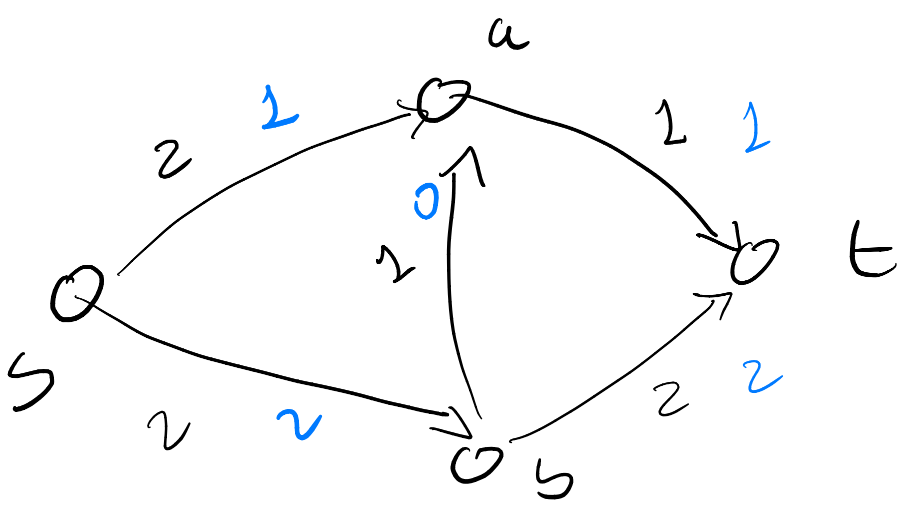
- $f_{u,v} = -f_{v,u}$

capacity
- $f_{u,v} \le c_{u,v}$ ($c_{u,v} = 0$ if $(u,v) \notin E$)

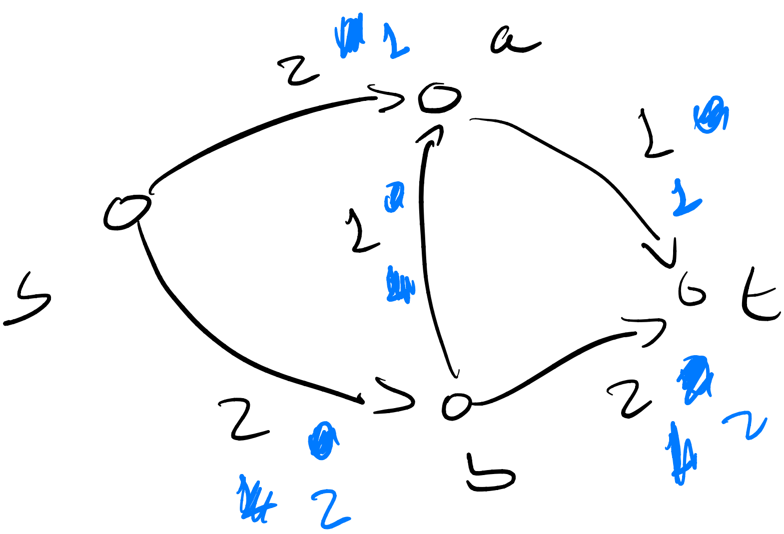
conservation
- $\forall v \quad v \neq s, \ v \neq t$
$$\sum_{w \in V} f_{vw} = 0$$



a
b
s
t
2
0
1/1
0 -2
0 -1
0 -1
1/1
2/2
2/1
0 -1

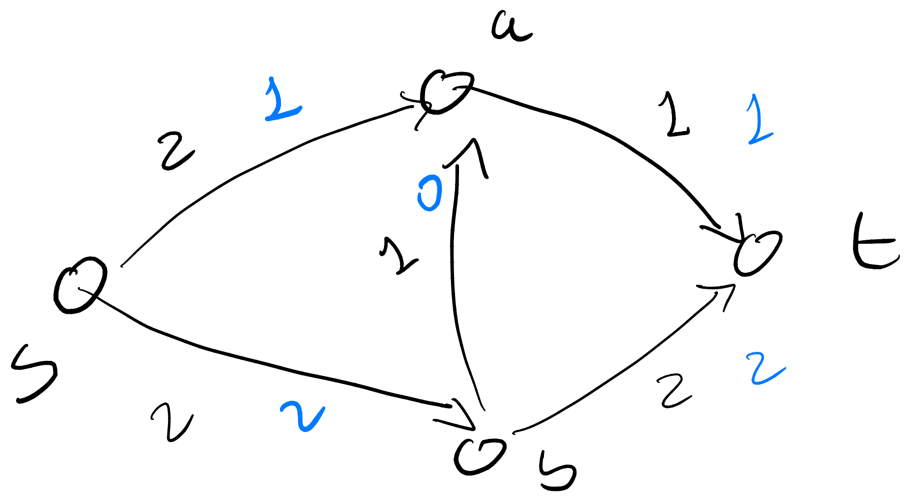


a
s
t
b
2 1
1 1
1 0
2 2
2 2



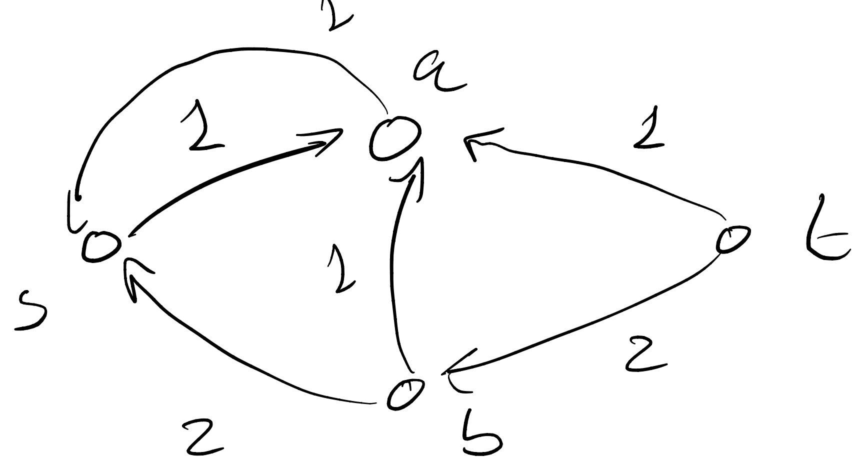
a
s
t
b
2 1
1 1
2 2
2 2

Residual Network Def.



Given a network N and a flow

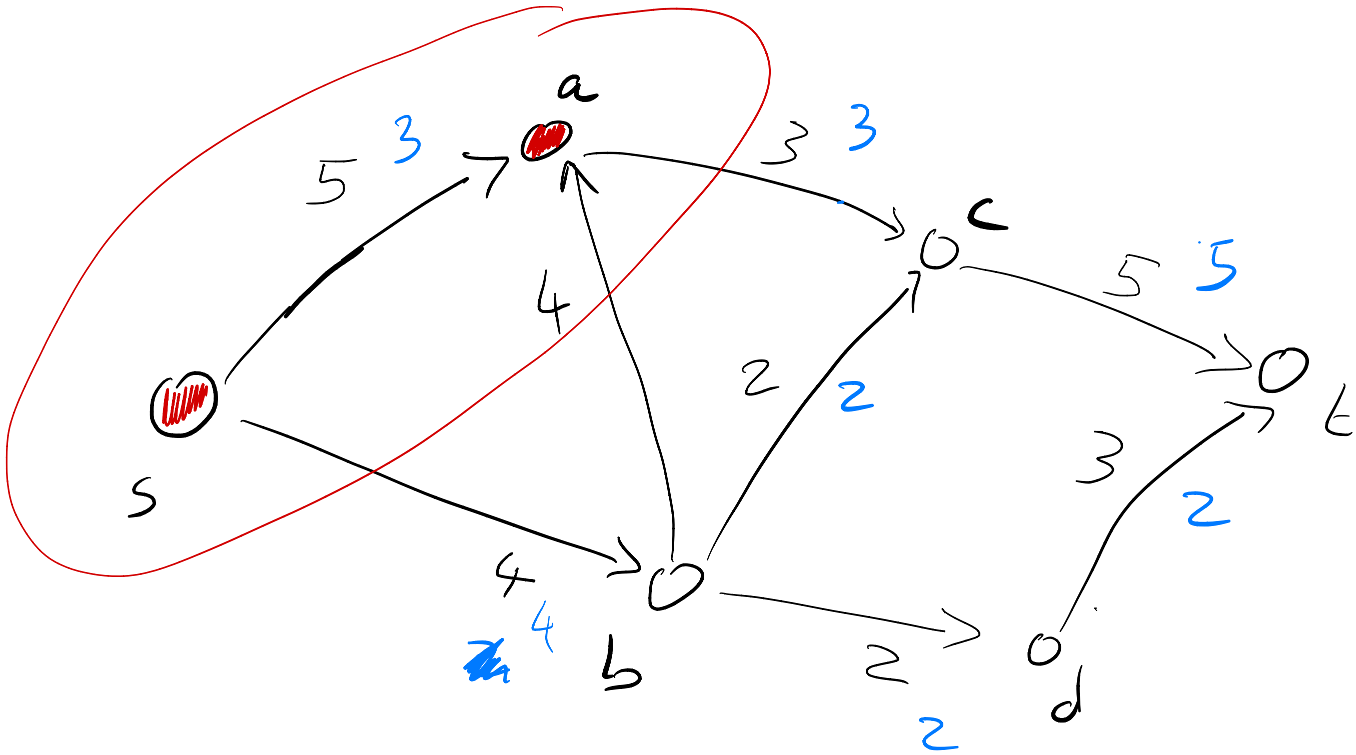
Residual network is network with same vertices of N and with edges for which $c_{uv} > f_{uv}$, labeled by $c_{uv} - f_{uv}$



Ford - Fulkerson Algorithm

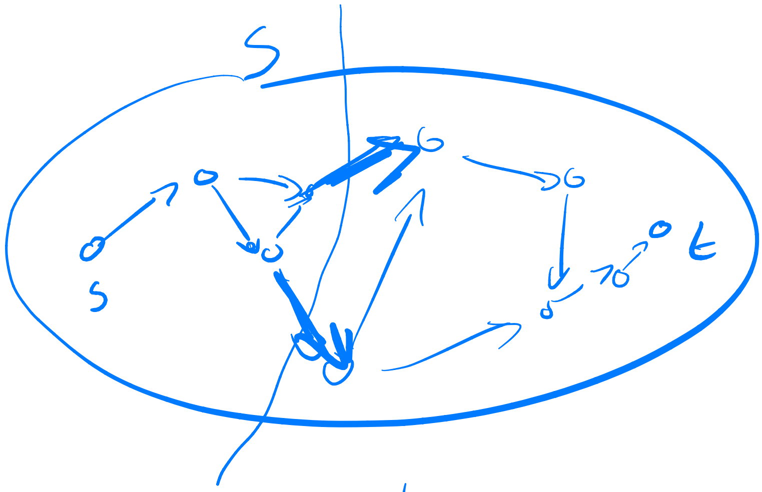
```
def FF(G, s, t, c):
    f = array indexed by all pairs
        u,v such that (u,v) or (v,u)
        is an edge, initialized to 0

    r = array indexed as before
        initialized to r[u,v] = c[u,v]

    while there is a path p
        from s to t made of edges
        (u,v) such that r[u,v] > 0:

        min = min(r[u,v]: (u,v) ∈ p)
        for each (u,v) ∈ p:
            f[u,v] = f[u,v] + min
            f[v,u] = f[v,u] - min
            r[u,v] = r[u,v] - min
            r[v,u] = r[v,u] + min

    return f
```

Example



$S = \{s, a\}$ $\quad$ $cap(S) = 3 + 4 = 7$

Cuts in a Network



Definition: a <u>cut</u> in a network is a subset S of vertices such that

$$s \in S$$
$$t \notin S$$

Def: the <u>capacity</u> of a cut S

is

$$\mathrm{cap}(S) = \sum_{a \in S} \sum_{b \notin S} c_{a,b}$$

Lemma Given a network N; For every flow f and every cut S

$$\mathrm{val}(f) \le \mathrm{cap}(S)$$

Lemma Given a network N;
For every flow f and every cut S

$$val(f) \leq cap(S)$$

Proof $val(f) = \sum_{v \in V} f_{sv}$ ← def of value of a flow

$$cap(S) = \sum_{a \in S} \sum_{b \notin S} f_{ab}$$

for every $v \neq s$, $v \neq t$

$$\sum_{w \in V} f_{vw} = 0$$ ← conservation constraint

$$\sum_{w \in V} f_{sw} = val(f)$$ ← def of value of flow

$$\sum_{v \in S} \sum_{w \in V} f_{vw}$$

$$= \underbrace{\sum_{w \in V} f_{sw}}_{val(f)} + \underbrace{\sum_{\substack{v \in S \\ v \neq s}} f_{vw}}_{=0} = val(f)$$

$$\mathrm{val}(f) = \sum_{v \in S} \sum_{w \in V} f_{vw}$$

$$= \cancel{\sum_{v \in S} \sum_{w \in S} f_{vw}} + \sum_{v \in S} \sum_{w \notin S} f_{vw}$$

$$= \sum_{v \in S} \sum_{w \notin S} f_{vw} \leq \sum_{v \in S} \sum_{w \notin S} c_{vw}$$

$$= \mathrm{cap}(S)$$

Theorem

Max Flow Mincut Theorem

Let N be a network
Let f be the output of Ford-Fulkerson
Let S be the set of vertices reachable from s in the residual network of N with respect to flow f

Then $cap(S) = val(f)$

For any other flow f' $val(f') \leq cap(S) = val(f)$
and so f is optimal