Shortest Paths
in graphs with negative
edge lengths

Compute Edit Distance
between two strings

$G = (V, E)$

$\ell(u,v)$ length of $(u,v) \in E$

there no negative length cycles

$s \in V$

Goal: find shortest paths from s to all other vertices

dist = array indexed by vertices
initialized to ∞

dist [s] = 0

$|V|-1$ for $i = 1$ to $|V|-1$:

$|E|$ for each edge $(u,v) \in E$:

$O(1)$ if dist [v] > dist[u] + $\ell(u,v)$:

dist [v] = dist [u] + $\ell(u,v)$

return dist

$O(|E|^2)$

$O(|V| \cdot |E|)$ $O(|V|^4)$

correctly computes distances from s to all vertices

Invariant

At end of step i of outer for loop
for every v

length of shortest path from s to v $\leq$ dist[v] $\leq$ length of s.p. from s to v among paths that use $\leq i$ edges

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If Invariant holds after step $|V|-1$

s.p. from s to v $\leq$ dist[v] $\leq$ s.p. from s to v that uses $\leq |V|-1$ edges

o—o—o—o—o—o = s.p. from s to v

dist[v] = s.p. from s to v

Invariant

At end of step i of outer for loop
for every v

length of shortest path from s to v $\leq$ dist[v] $\leq$ length of s.p. from s to v among paths that use $\leq i$ edges

$i=0$

$$dist[s]=0$$
$$dist[v]=\infty \quad v\neq s$$

✓

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invariant true after iteration k

$$dist_k[v] \leq d_k[v] \quad \forall v \in V$$

consider iteration $k+1$

$$dist_{k+1}[v] \leq dist_k[v]$$

length of shortest path from s to v with at most $k+1$ edges $= d_{k+1}(s,v)$

$$d_{k+1}(s,v) \leq d_k(s,v)$$

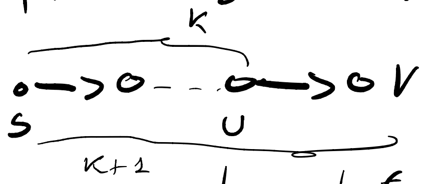
Case 1: $\quad d_{k+1}(s,v) = d_k(s,v)$

then $\quad dist_{k+1}[v] \leq dist_k[v]$

$$\leq d_k(s,v)$$
$$= d_{k+1}(s,v)$$

Case 2 $\quad d_{k+1}(s,v) < d_k(s,v)$

consider an optimal $(k+1)$-step path from s to v



call u vertex before v in opt path

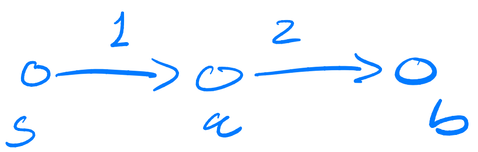
there is u: $\quad d_{k+1}(s,v) = d_k(s,u) + \ell(u,v)$

for every u: $\quad dist_{k+1}[v] \le dist_k[u] + \ell(u,v)$

$$
\begin{aligned}
dist_{k+1}[v] &\le dist_k[u] + \ell(u,v) && \text{update to dist in code} \\
&\le d_k(s,u) + \ell(u,v) && \text{invariant at step } k \\
&= d_{k+1}(s,v) && \text{defined } u \text{ as vertex before } v \text{ in opt } s \to v \text{ with } k+1 \text{ edges}
\end{aligned}
$$

$dist_{k+1}[v] \le d_{k+1}(s,v)$

$$dist_k[v] \cancel{\overset{?}{=}} d_k(s,v)$$



$i=0$	dist	0	∞	∞	
$i=1$		0	1	∞	s, a
		0	1	3	a, b

Dijkstra

dist $[v] = \infty$ $v \neq S$
dist $[S] = 0$
while Q not empty
  v = Q.delete_min()
  for all w: (v,w) ∈ E :
    update(dist, v, w)

DAG

dist $[v] = \infty$ $v \neq S$
dist $[S] = 0$
compute top. sort
for each v:
  for u: (u,v) ∈ E:
    update(dist, u, v)

General

dist $[v] = \infty$ $v \neq S$
dist $[S] = 0$
for i = 1 to $|V| - 1$
  for each (u,v) ∈ E:
    update(dist, u, v)

if dist $[v] >$ dist $[u] + \ell(u,v)$:
  dist $[v] =$ dist $[u] + \ell(u,v)$

→ update(dist, u, v)

Cormen Leiserson Rivest (Stein)

consider shortest from s to v

$$s \to v_1 \to v_2 \to \dots \to v_{k-1} \to v$$

	s	v_1	v_2	$\dots$	v_{k-1}	v
dist	0	∞	∞		∞	∞
	$\vdots$					
	0	$\le \ell(s,v_1)$				
	0	$\le \ell(s,v_1)$	$\le \ell(s,v_1) + \ell(v_1,v_2)$			

update( )

update( )

update(s, v_1)

$\vdots$

update(v_1, v_2)

$\vdots$

update(v_2, v_3)

$\vdots$

update(v_{k-1}, v)

shortest path from s to v $\le$ dist[v] $\le$ $\ell(s,v_1) + \ell(v_1,v_2) + \dots + \ell(v_{k-1}, v_k)$

= length of path $s \to v_1 \to \dots \to v_{k-1} \to v$

= shortest path from s to v

dist[v] = s.p. from s to v

All pairs shortest path

Given $G = (V, E)$

$\ell(u, v) \quad (u, v) \in E$

no negative cycles

Goal compute $d(a, b)$ for all $a, b \in V$

length of shortest path from a to b

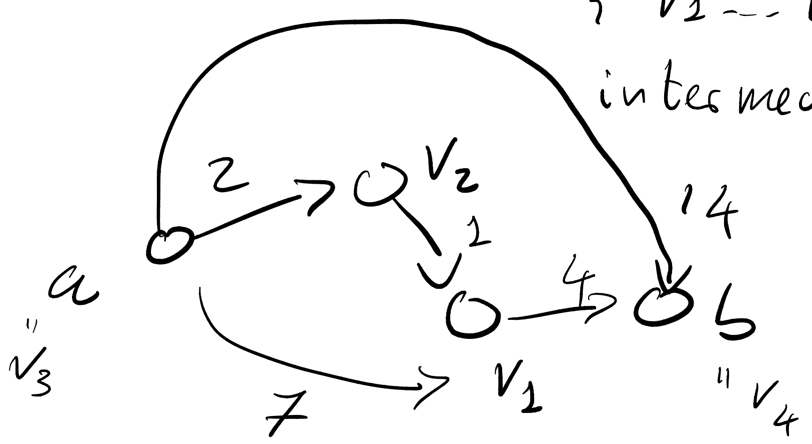
run previou alg for each $s \in V$

$$O(|V| \cdot |V| \cdot |E|) = O(|V|^2 \cdot |E|)$$

Call $V = \{v_1, v_2, v_3 \dots v_n\}$ $n = |V|$

Define for $k \geq 0$

$d_k(a,b)$ = length of shortest path from a to b among paths that use a subset of $\{v_1 \dots v_k\}$ as intermediate steps



$d_0(a,b)$ $= 14$

$d_1(a,b)$ $= 11$

$d_2(a,b)$ $= 7$

$$d_n(a,b) = d(a,b)$$

$$d_0(a,b) = \begin{cases} 0 & \text{if } a=b \\ \ell(a,b) & \text{if } (a,b)\in E \\ \infty & \text{otherwise} \end{cases}$$

$$d_{k+1}(a,b) = \min\{ d_k(a, v_{k+1}) + d_k(v_{k+1}, b), \; d_k(a,b) \}$$

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dist[k, a, b] = $d_k(a,b)$

Initialize $n \times n \times n$ array dist to ∞

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for each a, b ∈ V
    if a == b:
        dist[0, a, b] = 0
    else if (a, b) ∈ E:
        dist[0, a, b] = ℓ(a, b)
```

$O(|V|^2)$

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for k = 1 to n:                                        |V|
    for each a, b ∈ V:                                 |V|^2
        dist[k, a, b] = min(dist[k-1, a, b],           O(1)
                            dist[k-1, a, v_k] + dist[k-1, v_k, b])
return dist[n : :]
```

$O(|V|^3)$