Problem Set 4 Solutions

This problem set covers the material of lectures 20 and 21. This problem set is not due and it will not be graded, but you are welcome to come to office hours to discuss your solutions. In the second midterm and in the final you will not be tested on the material of lectures 22-24.

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1. Say that a decision problem C is *recognizable-complete* if it is recognizable and for every recognizable problem R we have $R \leq C$.

   (a) Prove that the Halting Problem is recognizable-complete.

   (b) Prove that if C is recognizable-complete then $\bar{C}$ is not recognizable.

   Solution. Let R be a recognizable problem and P_R be a program that recognizes R, meaning that for every x such that the correct answer for x with respect to problem R is YES, we have that $P_R(x)$ terminates and outputs YES, and for every x such that the correct answer is NO, we have that $P_R(x)$ either runs for ever or terminates and outputs NO. Let P'_R be a modification of program P_R such that $P'_R(x)$ simulates $P_R(x)$ and such that if the simulation halts with a NO answer then $P'_R(x)$ runs for ever.

   We reduce R to Halting in the following way: we map an input x of R to the input P'_R, x of the Halting problem. By construction, the answer for x with respect to problem R is YES if and only if $P'_R(x)$ halts.

   The above reduction can be defined for every recognizable problem R, proving that Halting is recognizable-complete.

   If C is recognizable-complete, then Halting reduces to C, and so $\overline{\text{Halting}}$ reduces to $\bar{C}$. If $\bar{C}$ was recognizable, then, because of the above reduction, $\overline{\text{Halting}}$ would be recognizable, but we proved that $\overline{\text{Halting}}$ is not recognizable, so $\bar{C}$ is not recognizable.

2. Consider the decision problem IOH (for *Infinitely Often Halting*) defined as follows. An input for IOH is a program P. The question is whether there are infinitely many different inputs x such that P halts on input x. Prove that IOH is not recognizable.

   [Hint: reduce from the complement of the Halting Problem]

Solution. Consider the following reduction. Given a program P and an input x for P, we define the following program P_x:

- Input: z
- $n = \text{length}(z)$
- Simulate $P(x)$ for up to n steps
- If $P(x)$ has not terminated after n steps, then stop
- Else run for ever

Let us study the set of inputs z on which P_x does and does not halt.

If $P(x)$ does not halt, then the simulation of $P(x)$ has not terminated after n steps, which means that $P_x(z)$ halts for every z and, in particular P_x halts for infinitely many inputs.

If $P(x)$ halts, call t the finite number of steps after which $P(x)$ halts. This means that for every z of length $\leq t-1$ the computation $P_x(z)$ halts, and for every z of length $\geq t$ $P_z(x)$ runs for every. Note that there is a finite number of binary strings z of length $\leq t-1$, and so, in this case, there is only a finite number of inputs on which P_x halts.

Because of the above reasoning, the mapping $P, x \to P_x$ is a reduction from $\overline{\text{Halting}}$ to IOH, proving that IOH is not recognizable.